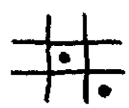
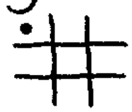
The checkers problem told to me by M.O. Rabin

Last week, during the annual seminar at the University of Newcastle-upon-Tyne, Michael O. Rabin told me the following problem.

Consider an infinite checkers board, of which the columns and rows are identified by the integer coordinates x and y respectively. Initially, there is a piece on each square whose coordinates satisfy $(\text{even}.x \equiv \text{even}.y) \wedge y \leq 0$. Pieces can be moved "upwards" by the usual "capturing moves": from  to  and from  to  . The question posed is whether there is an upper bound on the y-coordinates of the squares that may be occupied.

* * *

Because the question asked is about y-coordinates, we abstract for a moment from the x-coordinate. The two original moves then become one and the same move: from  to  . In order to capture that a high piece is created from ("is as good as", "corresponds to") its two successors — those who know him see Fibonacci lurking around the corner! —, we give a piece (on a square) at height y a weight φ^y with φ

the positive root of $\varphi^2 = \varphi + 1$. The equation is chosen so that the weight of the new piece equals the weight of the two pieces it replaces, i.e. a move is neutral as far as total weight is concerned. By restricting ourselves to the positive root, we keep all weights positive, i.e. total weight a monotonic function of the number of pieces involved. Solving the equation yields

$$\varphi = (1 + \sqrt{5})/2 \quad .$$

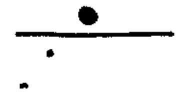
The fact that a move is neutral as far as total weight is concerned, means that the creation of a piece at height Y (with Y>0) uses from the original configuration a set of pieces with total weight φ^Y. This "target weight", being φ^Y and φ being positive, grows exponentially with Y. We shall next observe that, in view of the shape of the two moves —x re-enters the picture—, the original pieces involved in the creation of a piece at height Y come from a restricted area. Calling their total weight the "available weight", we shall show that the latter grows linearly with Y. Hence, the condition

target weight ≤ available weight

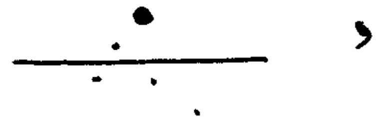
imposes an upper bound on Y: the answer to the question posed is "Yes".

In order to establish the available weight we observe the moves for small values of Y .

$Y = 1$ requires a single move, say

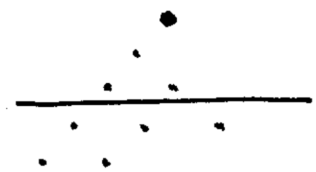


$Y = 2$ requires two moves, say



the one thin dot above the line indicating a square that has been temporarily occupied

$Y = 3$ requires two moves more.



For the restricted areas from which the original pieces have to be recruited we take infinite (truncated) triangles

$Y = 1$ $\qquad$ $Y = 2$ $\qquad$ $Y = 3$

etc. $\qquad$ etc $\qquad$ etc.

For $Y = 1$, the available weight $(\Sigma n: n \geq 0: (n+1) \cdot \varphi^{-n})$ is finite, and so is the increment $(\Sigma n: n \geq 0: \varphi^{-n})$. Summing the sequences one finds for the available weight

$$\frac{4 + 2\sqrt{5} + Y * (3 + \sqrt{5})}{2}$$

The smallest value for Y such that the target weight φ^{Y} exceeds the available weight is 7: then the target weight equals $(29 + 13\sqrt{5})/2$, the available weight is only $(25 + 9\sqrt{5})/2$.

For $Y=6$, the target weight $(18+8\sqrt{5})/2$ is less than the available weight $(22+8\sqrt{5})/2$, but this does not justify the conclusion that $Y=6$ is attainable. $Y=6$ can be achieved, but the only way of showing this that I know of is showing a game that does the job. The required game turns out to have 53 moves, which makes finding it and reporting it somewhat of a challenge. I could convince myself that $Y=6$ could be reached only

(i) after having realized that the constraint of at most 1 piece per square is inessential and can be dropped

(ii) after having decided to play the game backwards.

Below, we show successive stages of the backwards game: in the centre the configuration of the pieces, to the left the y-coordinates of the rows in question and —by way of check— to the right for each row the total number of pieces in it.

```
6           1          1

5         1            1
4       1              1

4       1   1          2
3             1

3     1   1   1        3
2   1   1              2
```

```
  2                   1   2   1   1                  5
  1                         1   1   1                3

  1                 1   1   2   2   2                8
  0               1   1       1   1   1              5

  0               2   2   2   2   3   2             13
 -1             1   1   2   1   1   1   1            8

  0               1   1   1   1   1   1              6
 -1             2   2   3   2   2   2   2           15
 -2           1   1   1   1   1       1   1          7

  0               1   1   1   1   1   1              6
 -1             1   1   1   1   1   1   1            7
 -2           2   2   2   2   2   1   2   2         15
 -3         1   1   1       1   1   1   1   1        8
```

and now not repeating all the constant rows

```
 -1             1   1   1   1   1   1   1            7
 -2           1   1   1   1   1   1   1   1          8
 -3         2   1   2   1   2   2   1   2   2       15
 -4       1           1   1   1   1       1   1      7

 -2           1   1   1   1   1   1   1   1          8
 -3         1   1   1   1   1   1   1   1   1        9
 -4       2       1   1   2   2   1   1   1   2     13
 -5     1       1       1   1       1           1    6

 -3         1   1   1   1   1   1   1   1   1        9
 -4       1   0   1   1   1   1   1   1   1   1      9
 -5     1   1   1       2   1   1   1       1   1   10
 -6           1       1           1       1          4

 -4       1   0   1   1   1   1   1   1   1   1      9
 -5     1   1   1       1   1   1   1       1   1    9
 -6           1       1   1       1       1          5
 -7                         1                        1
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There may exist a game of 51 moves, but I am not interested in that optimization. The above is already more elaborate than I had hoped.

Austin, 20 September 1992

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