## On arcs and angles

For the sake of comparison I shall give two presentations of (almost) the same theory. It is about a little corner of Euclidean geometry, relating the size of angles to the length of arcs of circles. When measuring the length of such arcs, we take the radius of the circle as our unit of length.

I shall first present the theory as I remember it from half a century ago.

Lemma 0 was about angles with their vertex at C (= the centre of the circle) and states that in fig. 0

fig. 0

$$\alpha = \beta \equiv \overset{\frown}{AA'} = \overset{\frown}{BB'}$$
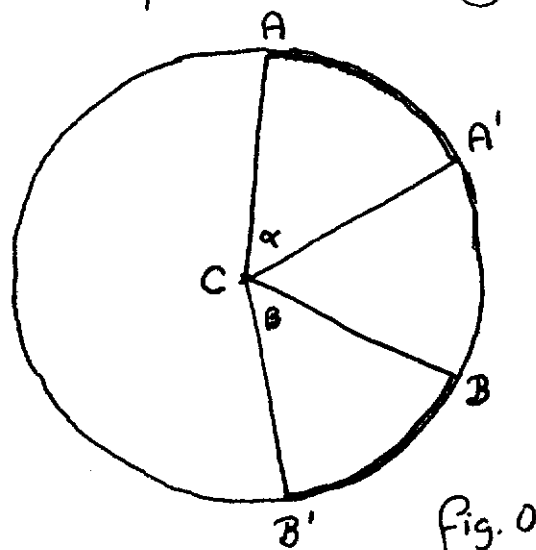
or "equal angles, equal arcs". This theorem was proved via the congruence $\triangle ACA' \cong \triangle BCB'$, and allows us to introduce the radian as unite of angle size: the size of an angle with vertex at C __equals__ the length of the corresponding arc. This last formulation I consider a rephrasing

of Lemma 0 : $\angle ACA' = \stackrel{\frown}{AA'}$

Lemma 1 was concerned with the size of $\angle APA'$ with all three points on the circle: it states

$$\angle APA' = \tfrac{1}{2} \stackrel{\frown}{AA'} \quad .$$

fig.1

The proof identifies two isosceles triangles and then establishes $\angle ACA' = 2 * \angle APA'$; with Lemma 0 , Lemma 1 then follows.
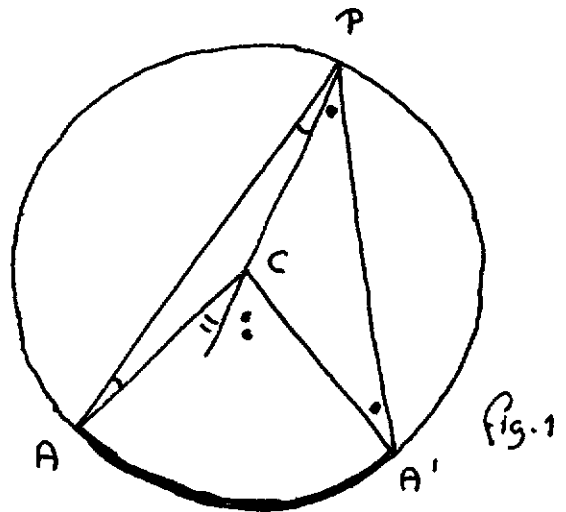
The mathematics is already getting slightly unattractive, for the argument based on fig.1 is not immediately applicable to fig.2 . This can be remedied by giving the analogous argument for fig.2 , but we are then still left with the question of whether fig.1 and fig.2 cover all cases.

fig. 2
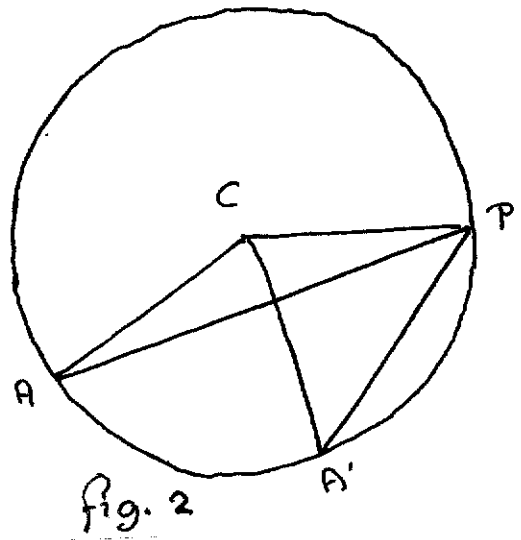
Lemma 2 deals with the case that the vertex is an arbitrary point P inside or on the circle, as in fig.3 : it states $\qquad \angle APB = \tfrac{1}{2} (\stackrel{\frown}{AB} + \stackrel{\frown}{A'B'})$.

The proof is by observing

$$\angle APB = \angle B'BA' + \angle BA'A$$

and then applying Lemma 1 twice. Here we don't struggle with the position of the circle's centre, but do face the question of P lying outside the circle:
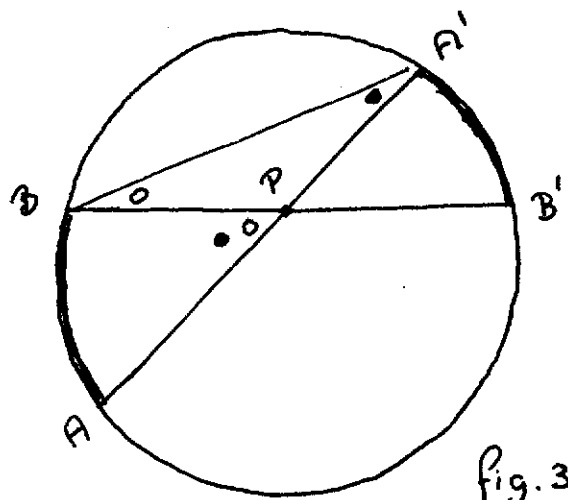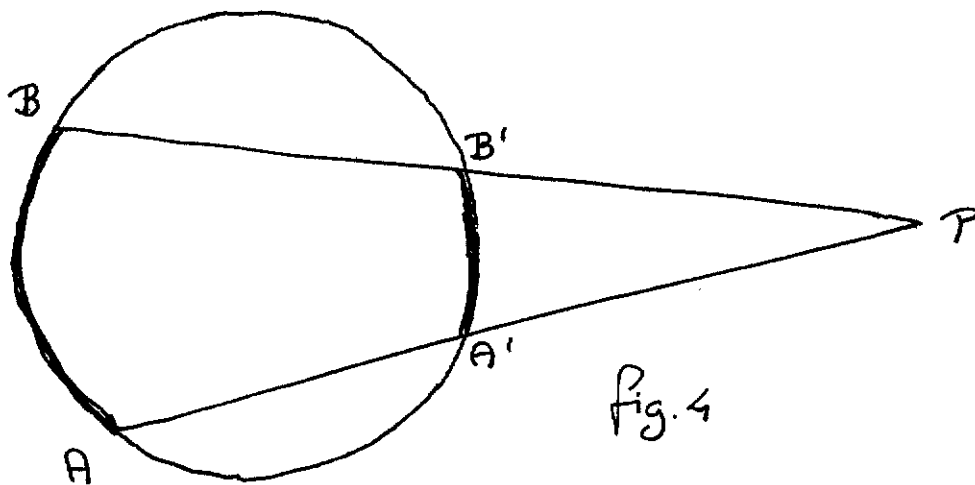

fig. 3

fig. 4

My highschool textbook solved this problem by introducing a new theorem — Lemma 3 — stating that in fig. 4

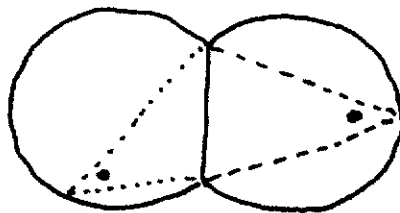$$\angle APB = \tfrac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{A'B'}) \quad,$$

and proving this theorem afresh (again by drawing the auxiliary line A'B).

3

The above is not too satisfactory for a variety of reasons.

- Lemmata 0 and 1 are special cases of Lemma 2
- Lemma 2 and 3 ask to be merged into a single lemma, e.g. by the introduction of something like a negative arc.
- There is no uniform, nonpictorial definition of which arcs have to be added or subtracted to yield which angle.

A closer inspection tells us that the notion of an angle, as used thus far, is probably too crude for our purposes. For instance, the traditional definition of a circle is the locus of all points at given distance from a given point; Lemma 2 invites the alternative definition as the locus of points from where a given line segment is seen under the same angle or something like that, but our definition should reject fig. 5 as our circle thus defined! (The vertical line is the segment in question.)

fig. 5

Our notion of "angle" as used thus far is based on the notion of the angle

between two rays, both
starting at the vertex,
as in fig. 6. It is a symmetric
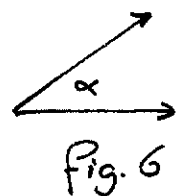function of the two rays that
delineate it.

Fig. 6

We now present an alternative. We don't
know its original inventor(s) but do know
that no one has promoted it more vigorous-
ly or explored it more extensively than
S-C Chou, X-S Gao, & J-Z Zhang, in their
book "Machine Proofs in Geometry" [0].

We tentatively call it "line angle"
because it associates an angle not with
two rays but with two full lines,
each of them extending in both directions
to infinity. (This is perhaps why Chou,
Gao, & Zhang speak of "full angles".)
We denote it - equally tentatively - by
the infix " $\neq$ ". It is not a symmetric
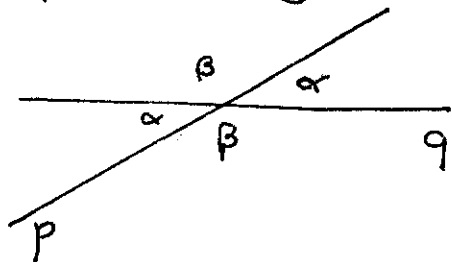function of its arguments: in fig. 7,

fig. 7

$p \neq q = \alpha$ and $q \neq p = \beta$. In words: $p \neq q$
is the angle over which p must be
rotated clockwise so as to make it

parallel to q . The picture in fig. 7 strongly suggests $p \not\angle q + q \not\angle p = \pi$, but that is an equality we shall return to later.

What we have gained can be seen by observing how the notion of the line angle does away with the anomaly signalled in fig. 5 . Consider two given points P and Q , and line p through P and line q through Q such that for some given α , $p \not\angle q = \alpha$; the locus of the intersection point of p and q is a circle through P and Q :

fig. 8
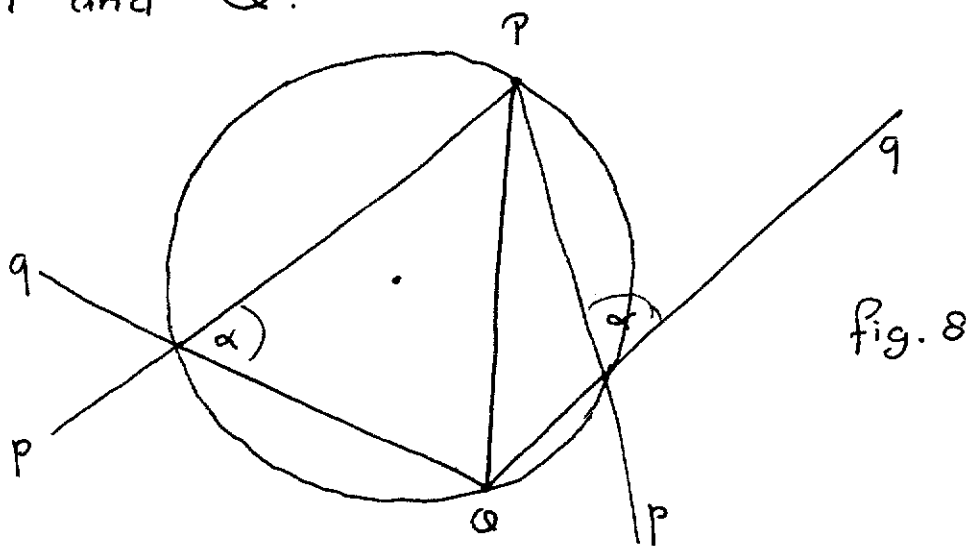
In short, the problem is solved by distinguishing the endpoints of the line segment — e.g. by naming them differently — and subsequently distinguishing the lines of a pair by the endpoint through which they pass. The convincing elegance reflected

in fig. 8 is strong evidence in favour of the notion of the line angle.

We now turn our attention to fig. 3 and fig. 4, and observe that in fig. 3, in which the arcs had to be added, the arcs from A to B and from A' to B' are both in the clockwise direction, whereas in fig. 4, where we had to take the difference of the arc lengths, one of the two — viz. from A' to B' — goes counter-clockwise. This observation invites the introduction of the notion of the directed arc, tentatively denoted by $\overset{\frown}{PQ}$, whose length is counted positively or negatively depending on whether the traversal of the arc from P to Q is clockwise or not. Directed arcs thus have the property

$$\overset{\frown}{PQ} = -\overset{\frown}{QP} \quad .$$

We still have to decide in which direction we count the arc length as positive. We decide that when the arc from P to Q goes in the clockwise direction, the value of $\overset{\frown}{PQ}$ is the (positive) length of that arc from P to Q.

Clocks being what they are, arcs $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$ in fig. 3 are both positive,

whereas arc $\overset{\frown}{A'B'}$ in fig.4 is negative. With these conventions, Lemma 2 and Lemma 3 become a single theorem:

Let line a intersect a circle in points A and A'; let line b intersect that same circle in points B and B'. Then

(0) $\qquad a \not\angle b = (\overset{\frown}{AB} + \overset{\frown}{A'B'})/2$.

In the above there is one issue I skimmed over. Its first manifestation is with the definition of the directed arc. The notation $\overset{\frown}{PQ}$ strongly suggests that for given P and Q the directed arc $\overset{\frown}{PQ}$ is defined, but the endpoints do not determine whether "the arc from P to Q" is traversed clockwise or counter-clockwise. To the question which direction we should choose, the proper answer is "either". The two options differ by 2π and we postulate that real values differing by an integer multiple of 2π correspond to the same directed arc.

fig.9
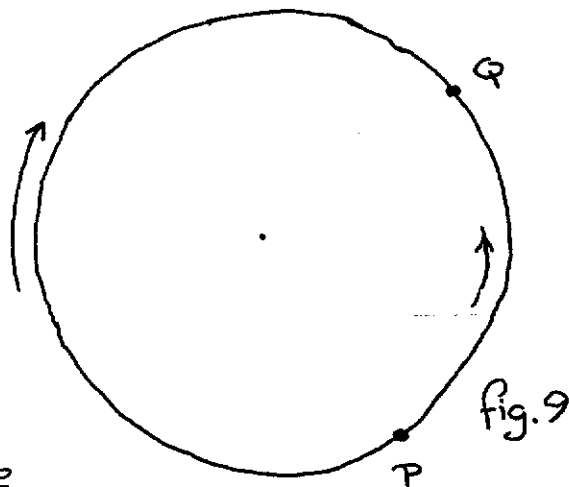
Its second manifestation is with the

formulation of the theorem: if line a intersects a circle, there are two points of intersection, which is A and which A'? The answer is again "either": the difference it makes in the sum

$$\widehat{A\vec{B}} + \widehat{A'\vec{B'}}$$

is, again, an integer multiple of 2π, i.e. ignorable. (Observe the following calculation, be it written down with some notational licence:

$$\widehat{A\vec{B}} + \widehat{A'\vec{B'}}$$

$$= \qquad \{ \widehat{P\vec{Q}} = -\widehat{Q\vec{P}} \}$$

$$(\widehat{A\vec{B}} + \widehat{B\vec{B'}}) + (\widehat{A'\vec{B'}} + \widehat{B'\vec{B}})$$

$$= \qquad \{ \widehat{P\vec{Q}} + \widehat{Q\vec{R}} = \widehat{P\vec{R}} \}$$

$$\widehat{A\vec{B'}} + \widehat{A'\vec{B}} \qquad\qquad . \qquad)$$

Its third manifestation is with the definition of the line angle: in fig.7 we defined $p \not\angle q$ to be equal to α, the clockwise rotation from p to q. But the alternative would have been a counter-clockwise rotation over β, or, equivalently, a clockwise rotation over $-\beta$. We observe $\pi - \beta = \alpha$, in general: real values that differ by an integer multiple of π represent the _same_

line angle. With directed arcs being defined up to multiples of 2π, formula (0) now makes perfect sense, thanks to the division by 2.
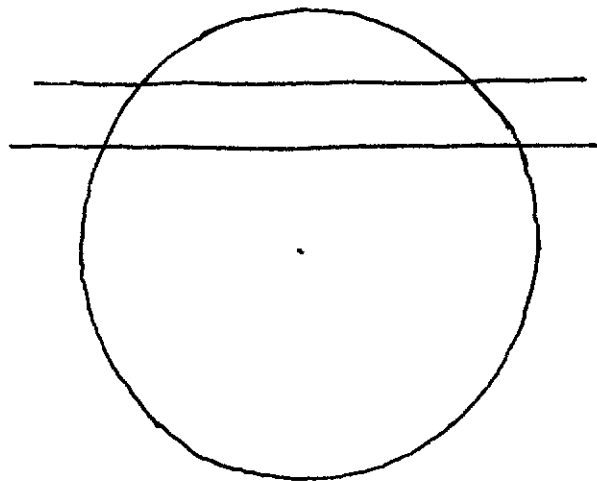

fig. 10

Finally, a single argument to prove lemmata 1, 2 and 3. Let us move, as in fig. 10, a line parallel to itself; the observation is that, for reasons of symmetry —and here you may be as explicit as you like— the one point of intersection moves in the counter-clockwise direction exactly as far as the other point of intersection does in the clockwise direction. Hence, if we move the line pair (a,b) parallel to it self, not only $a \neq b$ is constant, but also $\overgroup{AB} + \overgroup{A'B'}$. Hence, if (0) holds to begin with —e.g. when (a,b) intersect in C— , (0) continues to hold (as long as

both lines intersect the circle.

* * *

Having seen the alternative, the reader should realize how clumsy a theory I learned when I was young: when using Lemma 2 and/or Lemma 3, one either has to show that the intersection point lies inside/outside the circle, or one has to make a case analysis.

The point I wanted to make is that in designing a mathematical theory, the choice of concepts and definitions is crucial. I thought this was a very nice and simple example with which to drive the message home. Somehow, this note got longer than I had expected.

[0] "Machine Proofs in Geometry", Shang-Ching Chou, Xiao-Shan Gao, Jing-Zhong Zhang, World Scientific Publishing Co, Singapore, 1994

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