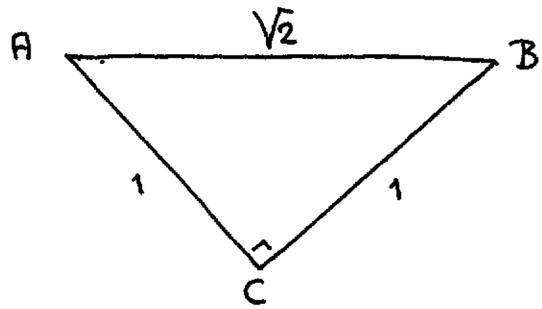
Z.P. Su's second problem

The other day Zhendong Patrick Su showed me the following problem of the last Putnam Competition, in which he had participated. (He had solved the problem, I did not.)

Consider the points in or on the right-angled triangle with hypothenuse of length $\sqrt{2}$:

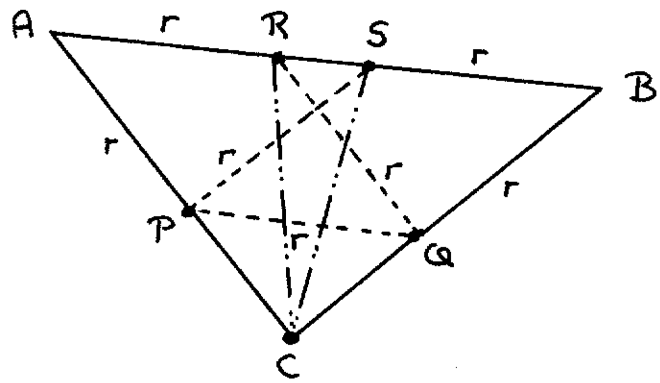


Let each of these points have 1 of 4 colours. Show the existence of a monochrome pair with mutual distance $\geq 2-\sqrt{2}$.

Let us call that critical distance r, i.e. $r = 2-\sqrt{2}$; let us call a monochrome pair at mutual distance $\geq r$ a "desired pair".

If A belongs to a desired pair, we are done; similarly for B. If neither A nor B belongs to a desired pair, we proceed as follows.

To begin with we observe that r satisfies $r = \sqrt{2} \cdot (1-r)$, from which we



conclude that, with $AP = r$ and $BQ = r$, we have $PQ = r$. Completing the diamonds with $AR = r$ and $BS = r$, we conclude $RQ = r$ and $PS = r$. Being the hypothenuse of a right-angled triangle with side $= r$, RC and SC are both $> r$.

In the case that neither A nor B belongs to a desired pair, A and B — more than r apart — are of different colour, and P, S, C, R, Q — all $\geq r$ apart from A and B — are all of the remaining 2 colours. Consequently, in the cyclic arrangement P, S, C, R, Q of these 5 points, 5 being odd, two successive points are of the same colour. Their mutual distance being at least r, they form a desired pair. QED.

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In the above argument we have appealed to the following theorem:

Consider a finite undirected graph in which each vertex is of degree 2. If each vertex is of one of 2 colours and the number of vertices is odd, there exists an edge that connects a vertex to a vertex of the same colour.

The proof is by double application of the pigeon-hole principle. Let the number of vertices —which equals the number of edges— be $2n+1$. Since there are 2 colours, there are at least $n+1$ vertices of the dominant colour; as they are all of degree 2, they give rise to $2n+2$ endpoints of the dominant colour. Since there are "only" $2n+1$ edges, at least 1 of them has 2 endpoints of the dominant colour.

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prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA