# A problem communicated by Laurens de Vries

In a recent letter, Netty van Gasteren wrote to me a theorem that Laurens de Vries had shown in Eindhoven, and for which he had seen a very complicated proof. I give the problem as stated:

To be proved

$$\langle \Sigma j:\ 1 \le j \le n:\ b.j \rangle =$$

$$\langle \Sigma j:\ 1 \le j \le n:\ b.j * \langle \Pi i:\ 1 \le i \le n \wedge i \ne j:\ 1 - \frac{b.i}{b.j} \rangle^{-1} \rangle \quad ,$$

when it has been given that

$$\langle \forall i,j:\ 1 \le i < j \le n:\ b.i \ne b.j \rangle \quad .$$

* * *

To simplify my calculations, I rephrased the theorem:

Let $W$ be a set of $n$ (by definition mutually different) numbers, and let $R$ satisfy

$$R = \langle \Sigma x:\ x \in W:\ \frac{x^n}{\langle \Pi y:\ y \in W - \{x\}:\ x - y \rangle} \rangle \quad ;$$

then $R = \langle \Sigma x:\ x \in W:\ x \rangle$

Leaving the verification of the cases $n \leq 1$ to the reader, we focus our attention on $n \geq 2$.

$R$ is defined as the sum of $n$ fractions, each with a numerator of degree $n$ and a denominator of degree $n-1$. Were we to reduce all of them to a common denominator, we would get the denominator

$$\langle \Pi\, x,y : x,y \in W \wedge x > y : x-y \rangle$$

of degree $n*(n-1)/2$ and a numerator of a degree 1 higher, but we first focus our attention to the sum of two of the fractions. For $p,q \in W \wedge p \neq q$, we observe

$$\frac{p^n}{\langle \Pi\, y : y \in W - \{p\} : p-y \rangle} + \frac{q^n}{\langle \Pi\, y : y \in W - \{q\} : q-y \rangle}$$

$=$ { with • $P.x = \langle \Pi\, y : y \in W - \{p,q\} : x-y \rangle$ }

$$\frac{p^n}{(p-q) * P.p} + \frac{q^n}{(q-p) * P.q}$$

$=$ { reduce to common denominator }

$$\frac{p^n * P.q \; - \; q^n * P.p}{(p-q) * P.p * P.q}$$

Because $P.x$ is a polynomial in $x$,

the numerator of our last fraction has a factor $p-q$, and hence we can eliminate the factor $p-q$ from both numerator and denominator. Since the other $n-2$ fractions have denominators that don't contain a factor $p-q$ we conclude that $R$ remains bounded when $p$ approaches $q$, hence $R$ can be written as a fraction from whose denominator the factor $p-q$ has disappeared. Since $p, q$ was an arbitrary pair of elements of $W$, we conclude that $R$ can be written as a fraction with a denominator of degree $0$ and hence a denominator of (at most) 1. For reasons of symmetry, we can conclude

$$(0) \qquad R = c * \langle \Sigma x : x \in W : x \rangle$$

for some value of $c$, which is "constant" in the sense that it does not depend on the values in $W$. In order to determine the value of $c$, we consider

$$\lim_{p \to \infty} (R/p) \qquad ,$$

where the other values in $W$ are kept constant. $R/p$ is the sum of $n$ fractions, $n-1$ have limit $0$, and one has the limit 1, hence $R/p$ has limit 1. (0) gives limit $c$, hence $c = 1$, and the theorem has been proved.

Remark I have used asymptotic techniques: values approaching 0 or $\infty$. With the same techniques I derived

$$\langle \Sigma x : x \in W : \frac{x^{n-1}}{\langle \Pi y : y \in W - \{x\} : x - y \rangle} \rangle = 1$$

$$\langle \Sigma x : x \in W : \frac{x^{n+1}}{\langle \Pi y : y \in W - \{x\} : x - y \rangle} \rangle =$$

$$\langle \Sigma x : x \in W : \langle \Sigma y : y \in W \wedge x \leq y : x \cdot y \rangle \rangle \quad .$$

For the last result I needed complex numbers, and I did not try higher exponents of $x$. (End of Remark.)

The above argument was constructed in the very early morning of 29 February 1996, during a mostly sleepless night at Seton Hospital.

Austin, 2 March 1996

prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA