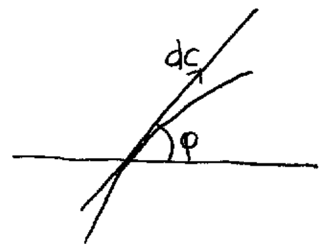
Ulrich Berger's argument rephrased

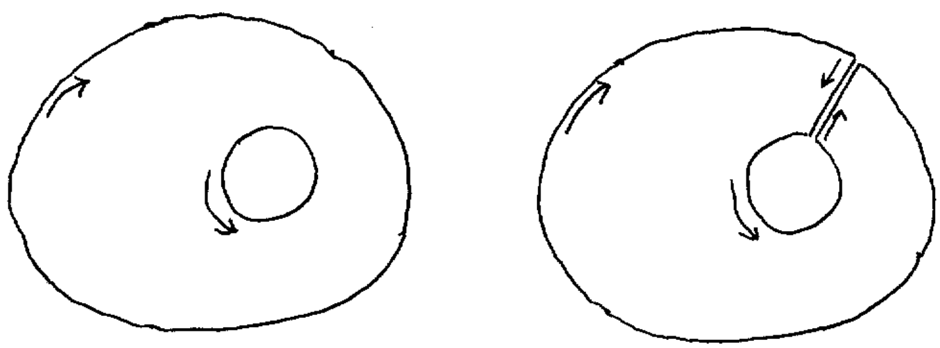
We introduce a special type of line integral, the "contour integral", in which the differential dc is an infinitesimal vector along the tangent. [With s the length along the contour, we have



$$dc = (\cos.\varphi\ ds\ ,\ \sin.\varphi\ ds).]$$

We only consider clockwise integrals along closed contours; hence the law $\oint dc = (0,0)$.

We now consider for a vector field f and some contour the contour integral $\oint f.(x,y) \bullet dc$, where $\bullet$ denotes the scalar product. We don't need to consider "bagels" with a hole in them



for the above two figures have equal contour integrals because the two contributions along the cut cancel.

Similarly, when we consider —like a jigsaw puzzle— a figure partitioned into a finite number

pieces, then the contour integral of the whole equals the sum of the contour integrals of the pieces. (We refer to this as "the Law".)

<u>Note</u> For $f.(x,y) = (y,0)$ or $f.(x,y) = (0,-x)$, the contour integral equals the area enclosed by the contour, and indeed: the area of the whole jigsaw puzzle equals the sum of the areas of the pieces. (End of Note.)

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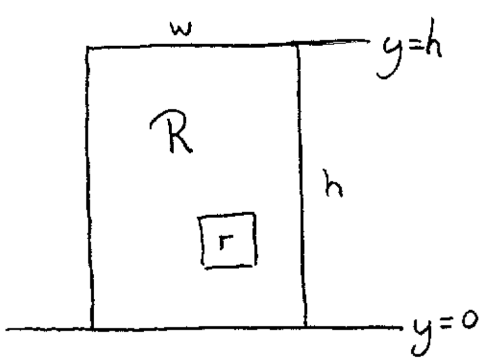
We only consider rectangles with horizontal and vertical sides. A rectangle is called <u>nice</u> when in at least one direction it has sides of integer length. The theorem to be proved is that in the case of a large rectangle R partioned in a finite number of little rectangles r, rectangle R is nice if all rectangles r are nice. We demonstrate that, with all rectangles r nice, R has an integer width w when its height h is noninteger. We do this by introducing an f



such that
(i) the contour integral of R equals w
(ii) the contour integral of each r is integer.
The Law then allows us to conclude that w, a sum of integers, is integer.

When we place R with its bottom at $y=0$ (and, hence, its top at $y=h$), some reflection shows that this is, for instance, realized by

$$f.(x,y) = (\underline{\text{if}}\ y \text{ is integer} \rightarrow 0\ [\!]\ y \text{ is noninteger} \rightarrow 1\ \underline{\text{fi}},\ 0) .$$

ad (i). Only the top side, of length w, contributes to the contour integral of R, 0 being integer and h not.
ad (ii). For a rectangle whose vertical sides are of integer length, the contributions along its horizontal sides (zero or not) cancel, and for a rectangle r whose horizontal sides are of integer length, the contributions along its horizontal sides (zero or not) are integer; in either case, nice rectangle r has a contour integral of integer value.

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