Q_1: Verify that the multivariate chain rule holds when
$z = x^2y + xy^2$, and $x = 3t$, $y = t^2$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + y^2)(3) + (x^2 + 2xy)(2t)$$

$$6xy + 3y^2 + (x^2(2t) + 2xy(2t)) \rightarrow 6t(t^2)(2t) \quad 12t^2(t^2) \quad 12t^4$$

$$6(3t)(t^2) + 3(t^2)^2 + ((3t)^2(2t) + 2(3t)(t^2)(2t))$$

$$3t^4 + 18t^3 + 18t^3 + 12t^4$$

$$= 36t^3 + 15t^4 \checkmark$$

∴ The multivariate chain rule theorem holds in both instances of differentiation.

$$z = (3t)^2(t^2) + (3t)(t^2)^2$$
$$= 9t^2(t^2) + (3t)(t^4)$$

$$[z]' = [9t^4 + 3t^5]' + 3t^5$$

$$z' = 36t^3 + 15t^4 \checkmark$$

Q_2: Use the Multivariate Chain Rule to find $\frac{dz}{dt}$ or $\frac{dw}{dt}$.
Let $z = xy^3 - x^2y$, $x = t^2+1$, $y = t^2-1$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dx}{dt}$$

$$= (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t)$$

$$= \underbrace{2t(y^3) - 2t(2xy)}_{\text{Part I}} + (2t)3xy^2 - (2t)x^2$$

$$= \boxed{8t^7 - 18t^5 - 4t^3 + 6t} \checkmark$$

$$\begin{array}{cccc} & t^2 & -1 & \\ t^2 & t^4 & -t^2 & \\ -1 & -t^2 & 1 & \\ & t^4 - 2t^2 + 1 & & \end{array}$$

$$\begin{array}{ccccc} & t^4 & -2t^2 & +1 \\ t^2 & t^6 & -2t^4 & +t^2 \\ -1 & -t^4 & 2t^2 & -1 \end{array}$$

$$2t(t^6 - 3t^4 + 3t^2 - 1)$$

$$\underbrace{2t^7 - 6t^5 + 6t^3 - 2t}$$

$$\begin{array}{cccc} & t^2 & -1 & \\ t^2 & t^4 & \cancel{-t^2} & \\ +1 & \cancel{t^2} & -1 & \end{array}$$

$$2(t^4 - 1)$$

$$2t(2t^4 - 2)$$

$$\underbrace{-4t^5 + 4t}$$

$$\begin{array}{cccc} & t^2 & +1 & \\ t^2 & t^4 & t^2 & \\ t+1 & t^2 & 1 & \end{array}$$

$$(t^4 + 2t^2 + 1)2t$$

$$2t^5 + 4t^3 + 2t$$

$$\begin{array}{cccc} t^2 & t^4 & -2t^2 & +1 \\ t^2 & t^6 & -2t^4 & +t^2 \\ t+1 & t^4 & -2t^2 & +1 \end{array} \Bigg] \rightarrow (t^6 - t^4 - t^2 + 1)3 = (3t^6 - 3t^4 - 3t^2 + 3)(2t)$$

$$6t^7 - 6t^5 - 6t^3 + 6t$$

$$6t^7 - 6t^5 \cancel{6t^3} + \cancel{6t} - 2t^5 - \cancel{4t^3} - \cancel{2t}$$

$$\underbrace{6t^7 - 8t^5 - 10t^3 + 4t}_{\text{Part II}}$$

$$2t^7 - 6t^5 + 6t^3 \cancel{-2t} - 4t^5 \cancel{+4t}$$

$$\underbrace{2t^7 - 10t^5 + 6t^3 + 2t}_{\text{Part I}} + \underbrace{6t^7 - 8t^5 - 10t^3 + 4t}_{\text{Part II}}$$

Q_3: $z = \sin x \cos y$, $x = \sqrt{t}$, $y = \frac{1}{t}$ $\begin{smallmatrix} f \\ g \end{smallmatrix}$

$$\frac{f'g - g'f}{g^2} \qquad -\frac{1}{(t)^2}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (\cos(x)\cos(y))\left(\frac{1}{2\sqrt{t}}\right) + (-\sin(x)\sin(y))\left(-\frac{1}{t^2}\right)$$

$$= \boxed{\frac{\cos(x)\cos(y)}{2\sqrt{t}} + \frac{\sin(x)\sin(y)}{t^2}} \checkmark$$

Q_7: $w = xe^{\frac{y}{z}}$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$(2t) \cdot$$