Find the radius of convergence and interval of convergence of the power series.

1. $\sum_{n=1}^{\infty} \frac{1}{n} \cdot x^n$

Let $a_n = \frac{x^n}{n}$

$a_{n+1} = \frac{x^{n+1}}{n+1}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}\, n}{(n+1)\, x^n}\right| = \lim_{n\to\infty}\left|\frac{x \cdot n}{n+1}\right|$

$= \lim_{n\to\infty}\left|\frac{x \cdot \cancel{n}}{\cancel{n}(1+\frac{1}{n})}\right|$

$-1 \le x < 1$ $\quad \boxed{[-1, 1)}$

$= \lim_{n\to\infty}\left|\frac{x}{1+\frac{1}{n}}\right| = |x| < 1$

Test @ $x = 1$

$\sum_{n=1}^{\infty} \frac{1^n}{n} \equiv \sum_{n=1}^{\infty} \frac{1}{n}$ $\therefore$ Diverges by p-test $\quad \boxed{R=1}$

Test @ (-1)

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$

AST: Let $U_n = \frac{1}{n}$

Decreasing ✓ $\rightarrow \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\therefore$ Converges by AST @ $x = -1$

2. $\sum_{n=1}^{\infty} \sqrt{n}\, x^n$

Let $a_n = \sqrt{n}\, x^n$

$\boxed{R=1}$

$a_{n+1} = \sqrt{n+1}\, x^{n+1}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\, x^{n+1}}{\sqrt{n}\, x^n}\right|$

$-1 < x < 1 \rightarrow \boxed{(-1, 1)}$

$= \lim_{n\to\infty}\left|\sqrt{\frac{n+1}{n}} \cdot x\right|$

Test @ 1

$\sum_{n=1}^{\infty} \sqrt{n} \cdot 1^n \rightarrow \sum_{n=1}^{\infty} \sqrt{n}$

$= \lim_{n\to\infty}\left|\sqrt{\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}}} \cdot x\right|$

TD Test:

$= \lim_{n\to\infty}\left|\sqrt{1+\frac{1}{n}} \cdot x\right| = |x| < 1$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{1/2} = \infty \neq 0$

$\therefore$ This diverges by Test for divergence which also applies for (-1)

3. $\sum_{n=1}^{\infty} \frac{n}{5^n} x^n$

Let $a_n = \frac{n}{5^n} x^n$

$a_{n+1} = \frac{n+1}{5^{n+1}} \cdot x^{n+1}$

$\boxed{R=5}$

$\lim_{n\to\infty}\left|\frac{(n+1)(x)^{n+1}\, 5^n}{(5)^{n+1}\, n \cdot x^n}\right| = \lim_{n\to\infty}\left|\frac{x \cdot (n+1)}{5n}\right|$

$-5 < x < 5$ $\quad \boxed{(-5, 5)}$

$= \lim_{n\to\infty}\left|\frac{\cancel{n} \cdot x(1+\frac{1}{n})}{5\cancel{n}}\right|$

Test @ 5

$= \lim_{n\to\infty}\left|\frac{x(1+\frac{1}{n})}{5}\right| = \frac{|x|}{5} < 1$

$\sum_{n=1}^{\infty} \frac{n}{\cancel{5^n}} \cdot \cancel{5^n} = \sum_{n=1}^{\infty} n$

NTT:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$

$\therefore$ Diverges by n^{th} term test for both (± 5)

4. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$

Let $a_n = \frac{x^n}{n \cdot 3^n}$

$a_{n+1} = \frac{x^{n+1}}{(n+1) 3^{n+1}}$

$\lim_{n\to\infty}\left|\frac{x^{n+1}\, n \cdot 3^n}{(n+1) 3^{n+1}\, x^n}\right| = \lim_{n\to\infty}\left|\frac{x \cdot n}{(n+1) \cdot 3}\right|$

$\boxed{R=3}$

$-3 \le x < 3$ $\quad \boxed{[-3, 3)}$

$= \lim_{n\to\infty}\left|\frac{x \cdot n}{3n+3}\right|$

$= \lim_{n\to\infty}\left|\frac{x \cdot \cancel{n}}{\cancel{n}(3+\frac{3}{n})}\right|$

Test @ $x = 3$

$\sum_{n=1}^{\infty} \frac{\cancel{3^n}}{n \cdot \cancel{3^n}} = \sum_{n=1}^{\infty} \frac{1}{n}$

$= \lim_{n\to\infty}\left|\frac{x}{3+\frac{3}{n}}\right| = \frac{|x|}{3} < 1$

$\therefore$ Diverges by p-test @ $x = 3$

Test @ -3

$\sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (3)^n}{n (3)^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

AST: Let $U_n = \frac{1}{n}$ $\quad \therefore$ Converges by AST at $x = -3$

Decreasing ✓

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

5. $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$

Let $a_n = \frac{x^n}{2n-1}$

$a_{n+1} = \frac{x^{n+1}}{2(n+1)-1} = \frac{x^{n+1}}{2n+1}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x)^{n+1}(2n-1)}{(2n+1)(x^n)}\right|$

$\boxed{R=1} \rightarrow -1 < x < 1$

Test @ $x = 1$ $\quad \boxed{[-1, 1)}$

$= \lim_{n\to\infty}\left|\frac{x \cdot (2n-1)}{2n+1}\right|$

$\sum_{n=1}^{\infty} \frac{1^n}{2n-1} \equiv \sum_{n=1}^{\infty} \frac{1}{2n-1}$

$= \lim_{n\to\infty}\left|x \cdot \frac{\cancel{n}(2-\frac{1}{n})}{\cancel{n}(2+\frac{1}{n})}\right|$

$= \lim_{n\to\infty}\left|x \cdot \frac{2-\frac{1}{n}}{2+\frac{1}{n}}\right| = |x| < 1$

LCT: Let $a_n = \frac{1}{2n-1}$

$b_n = \frac{1}{n}$

$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \frac{n}{2n-1} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} = C$

$\therefore$ Since $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ and $\sum b_n$ diverges by p-series, then $\sum a_n$ diverges as well by the conditions of the LCT.

$\therefore$ Series diverges at $x = 1$.

Test @ -1

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \equiv \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n-1}$

AST: Let $U_n = \frac{1}{2n-1}$

Decreasing ✓

$\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\}$

$\therefore$ This series converges by AST and converges at $x = -1$ endpnt.

$\lim_{n\to\infty} \frac{1}{2n-1} = 0$ ✓

6. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Let $a_n = \frac{x^n}{n!}$

$a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1} \cdot n!}{(n+1)!\, x^n}\right| = \lim_{n=\infty}\left|\frac{x \cdot n!}{(n+1)!}\right|$

$\therefore R = \infty$

$= \lim_{n=\infty}\left|\frac{x}{n+1}\right| = 0 < 1 \;\forall x$

$\rightarrow (-\infty, \infty)$

7. $\sum_{n=1}^{\infty} \frac{x^n}{n^4\, 4^n}$

Let $a_n = \frac{x^n}{n^4\, 4^n}$

$a_{n+1} = \frac{x^{n+1}}{(n+1)^4\, 4^{n+1}}$

$\lim_{n\to\infty}\left|\frac{n^4\, 4^n\, x^{n+1}}{(n+1)^4\, 4^{n+1}\, x^n}\right| = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^4 \cdot \frac{x}{4}\right|$

$\boxed{R=4}$ $\quad \boxed{[-4, 4]}$

$= \lim_{n\to\infty}\left|\left(\frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}\right)^4 \cdot \frac{x}{4}\right|$

$-4 < x < 4$

$= \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}} \cdot \frac{x}{4}\right| = \frac{|x|}{4} < 1$

$= |x| < 4$

Test @ $x = 4$

$\sum_{n=1}^{\infty} \frac{\cancel{4^n}}{n^4 \cdot \cancel{4^n}} = \sum_{n=1}^{\infty} \frac{1}{n^4}$

$\therefore$ Converges by p-series at $x = 4$.

Test @ $x = -4$

$\sum_{n=1}^{\infty} \frac{(-4)^n}{n^4 \cdot 4^n} \equiv \sum_{n=1}^{\infty} \frac{(-1)^n \cdot \cancel{4^n}}{n^4 \cdot \cancel{4^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

AST: Let $U_n = \frac{1}{n^4}$

Decreasing? ✓

$\{1, \frac{1}{16}, \frac{1}{81}, \frac{1}{256}\}$

$\lim_{n\to\infty} \frac{1}{n^4} = 0$ ✓

$\therefore$ This converges by AST when $x = -4$.