Q_1: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+8}}{\sqrt[4]{n}}$$

is absolutely convergent, conditionally convergent, or divergent.

Test for absolute convergence

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+8}}{\sqrt[4]{n}}\right| = \frac{1}{\sqrt[4]{n}} = \frac{1}{n^{1/4}} \therefore \text{ diverges via p-test since } p<1.$$

Condition #2 Test: Alternating Series Test

$$\frac{1}{(n+1)^{\frac{1}{4}}} \le \frac{1}{(n)^{1/4}} \checkmark$$

$$\lim_{n\to\infty} \frac{1}{(n)^{1/4}} = 0 \checkmark$$

∴ The series converges

∴ The series is conditionally convergent due to failing the test for absolute convergence theorem.

Q_2: Which of the following properties does the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2n^2+5}$$

have?

Test for alternating series:

$$\sum_{n=1}^{\infty} \frac{n}{2n^2+5} = \left\{\frac{1}{7}, \frac{2}{13}, \frac{3}{23}, \frac{4}{37}\right\} \checkmark$$

$$\lim_{n\to\infty} \frac{n}{2n^2+5} \overset{L.H}{=} \lim_{n\to\infty} \frac{1}{4n} = 0 \checkmark$$

Lim Comp. Test of $\sum a_n$ and $\sum b_n$

$\sum \frac{1}{2n} = \sum b_n$ and $\sum a_n = \sum_{n=1}^{\infty} \frac{n}{2n^2+5}$

$$\lim_{n\to\infty}\left|\frac{\frac{n}{2n^2+5}}{\frac{1}{2n}}\right| = \lim_{n\to\infty} \frac{2n^2}{2n^2+5} \overset{L.H}{=} \lim_{n\to\infty} \frac{\cancel{4n}}{\cancel{4n}} = \lim_{n\to\infty} 1 = \boxed{1}$$

Since $\sum b_n$ converges by p-test then by LCT [since $\lim_{n\to\infty} \frac{a_n}{b_n} = C$. ∴ $\sum a_n$ diverges.

∴ The series conditionally converges since the alternating series test converges while the limit comparison diverges when testing for absolute / conditional.

Q_3: Which one of the following properties does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}$$

have?

AST:

$$\lim_{n\to\infty} \frac{1}{3n+1} = 0 \checkmark$$

$$\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \ldots \checkmark$$

∴ $\sum U_n$ converges by AST

Limit Comp. Test

Let $\sum a_n = \sum U_n$

$\sum b_n = \frac{1}{3n}$ (diverges by p-test)

$$\lim_{n\to\infty}\left|\frac{\frac{1}{3n+1}}{\frac{1}{3n}}\right| = \lim_{n\to\infty} \frac{3n}{3n+1} \overset{L.H}{=} \lim_{n\to\infty} \frac{3}{3} = \boxed{1}$$

∴ Since $\sum b_n$ diverges by p-test and when $\sum b_n$ diverges then $\sum a_n$ diverges by the conditions of LCT

∴ This series conditionally converges due to the divergence in LCT, but converges by the conditions of AST.

Q_4: Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^{m-1} \frac{3}{\sqrt{1+m^2}}$$

is absolutely convergent, conditionally convergent, or divergent.

AST:

$$\lim_{m\to\infty} \frac{3}{\sqrt{1+m^2}} = 0 \checkmark$$

$$\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{5}}, \frac{3}{\sqrt{10}}, \frac{3}{\sqrt{17}}, \ldots \checkmark$$

Converges.

DCT:

Let $\sum a_n = \frac{3}{\sqrt{1+m^2}}$ — If $\sum b_n$ is divergent and $a_n \ge b_n$ then $\sum A_n$ also diverges.

$\sum b_n = \frac{1}{m}$

We know that $\frac{3}{\sqrt{1+m^2}} \ge \frac{1}{m}$ and $\sum b_n$ diverges by p-test

∴ Based on the conditions of DCT, $\sum a_n$ diverges.

∴ This series conditionally converges due to divergence by DCT, but convergence by AST (The condition(s)).

Q_5: Determine whether the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln(n)}$$

AST: $\lim_{n\to\infty} \frac{n}{\ln(n)} \overset{L.H}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty \neq 0$

∴ Diverges

DCT:

Let $\sum a_n = \frac{n}{\ln(n)}$

$\sum b_n = \frac{1}{n}$

We know

$\sum a_n \ge \sum b_n$

and

$\sum b_n$ diverges by p-test

∴ Series diverges as both tests diverge.

Q_6: Which of the following properties does the series

$$\sum_{n=1}^{\infty} \frac{n(-8)^n}{5^{n-1}}$$

have?

$$\sum a_n = \sum_{n=1}^{\infty} \frac{n(-8)^n}{5^{n-1}}$$

$$\sum a_{n+1} = \sum_{n=1}^{\infty} \frac{(n+1)(-8)^{n+1}}{5^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(-8)^{n+1}(5)^{n-1}}{5^n\, n\, (-8)^n}\right|$$

$$= \left|\frac{(n+1)\cancel{(-8)^n}(-8)^1 \cancel{(5)^n}(5)^{-1}}{\cancel{5^n}\cdot n \cdot \cancel{(-8)^n}}\right|$$

$$= \left|\frac{-8n-8}{5n}\right|$$

$$\lim_{n\to\infty}\left|\frac{-8n-8}{5n}\right| \xrightarrow{L.H} \lim_{n\to\infty}\left|\frac{-8}{5}\right| = \boxed{\frac{8}{5}}$$

Q_7: Determine whether the series

$$\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!}$$

is absolutely convergent, conditionally convergent, or divergent.

$$\text{Let } \sum a_n = \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!}$$

$$\sum a_{n+1} = \sum_{n=0}^{\infty} \frac{(-4)^{n+1}}{(2n+2)!}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(-4)^{n+1}}{(2n+2)!}}{\frac{(-4)^n}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(-4)^n}(-4)\cdot(2n)!}{\cancel{(-4)^n}(2n+2)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-4)(2n)!}{(2n+2)!}\right| = \lim_{n\to\infty}\left|\frac{(-4)\cancel{(2n)!}}{(2n+1)(2n+2)\cancel{(2n)!}}\right|$$

$$\lim_{n\to\infty}\left|\frac{-4}{(2n+1)(2n+2)}\right| = \boxed{0}$$

Since $\alpha<1$, this series absolutely converges by ratio test

Q_8: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^n}{(3n+1)2^{2n+1}}$$

$$\sum a_n = \sum_{n=1}^{\infty} \frac{\cancel{2^n}}{(3n+1)2^{2n}2} = \frac{1}{(3n+1)2^{n+1}}$$

$$\sum a_{n+1} = \sum_{n=1}^{\infty} \frac{1}{(3n+4)2^{n+2}}$$

$$\frac{\cancel{2^n}\,\cancel{2^1}}{\cancel{2^n}\,2^2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{(3n+1)2^{n+1}}{(3n+4)2^{n+2}}$$

$$\lim_{n\to\infty}\left|\frac{3n+1}{(3n+4)2}\right| = \lim_{n\to\infty}\left|\frac{3n+1}{12n+8}\right| \overset{L.H}{=} \lim_{n\to\infty}\left|\frac{3}{12}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{4}\right| = \frac{1}{4} < 1$$

∴ Series absolutely converges.

Q_9: Determine whether the following series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{8^n}$$

is absolutely convergent, conditionally convergent, or divergent.

Let $\sum a_n = \frac{n!}{8^n}$

$\sum a_{n+1} = \frac{(n+1)!}{8^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{\cancel{n!}\,\cancel{8^n}\,8^1}{\cancel{8^n}(n+1)\cancel{!}}\right| \text{ flipped } \quad \frac{\cancel{n}\cdot\cancel{(n-1)}\cdot\cancel{(n-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{(n+1)\cdot\cancel{(n)}\cdot\cancel{(n-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$\lim_{n\to\infty}\left|\frac{n+1}{8}\right| = \infty > 1$$

∴ This series diverges by ratio test

Q_{10}: Decide whether the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 5^n$$

converges or diverges.

$$\frac{(2n)\cdot(2n-1)\cdots 3\cdot 2\cdot 1}{(2n+2)\cdot(2n+1)\cdot(2n)\cdots 3\cdot 2\cdot 1}$$

$$\text{Let } \sum a_n = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 5^n$$

$$\sum a_{n+1} = \sum_{n=1}^{\infty} \frac{((n+1)!)^2 5^n}{(2n+2)!}$$

$$\frac{(n+1)^2\cdot\cancel{(n)^2}\cdot\cancel{(n-1)^2}\cdots\cancel{(3)^2}\cancel{(2)^2}\cancel{(1)^2}}{\cancel{(n)^2}\cdot\cancel{(n-1)^2}\cdots\cancel{(3)^2}\cdot\cancel{(2)^2}\cdot\cancel{(1)^2}}$$

$$\lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2 5^{n+1}}{(2n+2)!}}{\frac{(n!)^2 5^n}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{((n+1)!)^2\, 5^{n+1}(2n!)}{(2n+2)(n!)^2\, 5^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2\cdot 5}{(2n+2)(2n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{5n^2+10n+5}{4n^2+6n+2}\right|$$

$$L.H \lim_{n\to\infty}\left|\frac{10n+10}{8n+6}\right|$$

$$L.H \lim_{n\to\infty}\left|\frac{10}{8}\right| = \boxed{\frac{5}{4}} > 1$$

$(n+1)(n+1) \to n^2+2n+1$

$(2n+2)(2n+1) = 4n^2+6n+2$

∴ This series diverges by ratio test.

Q_{11}: Determine whether the following series

$$\sum_{n=1}^{\infty} \frac{3n+4}{(2n)!}$$

is absolutely convergent, conditionally convergent, or divergent.

Let $a_n = \frac{3n+4}{(2n)!}$

$$\frac{\cancel{(2n)}\cdot\cancel{(2n-1)}\cdot\cancel{(2n-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{(2n+2)\cdot(2n+1)\cdot\cancel{(2n)}\cdots\cancel{2}\cdot\cancel{1}}$$

$a_{n+1} = \frac{3n+7}{(2n+2)!}$

$$\lim_{n\to\infty}\left|\frac{\frac{3n+7}{(2n+2)!}}{\frac{3n+4}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{(3n+7)(2n)!}{(2n+2)!\,(3n+4)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3n+7}{(2n+2)(2n+1)(3n+4)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3n+7}{12n^3+28n^2+22n+8}\right|$$

$$L.H \lim_{n\to\infty}\left|\frac{3}{36n^2+56n+22}\right| = \boxed{0} = \frac{3}{\infty}$$

$(2n+2)(2n+1) = 4n^2+6n+2$; $(4n^2+4n+2)(3n+4) = 12n^3+12n^2+6n+16n^2+16n+8$

∴ This series converges by the ratio test.

Q_{12}: Determine whether the following series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{4n^2+5}$$

is absolutely convergent, conditionally convergent, or divergent.

Let $a_n = \frac{2^n}{4n^2+5}$

$a_{n+1} = \frac{2^{n+1}}{4(n+1)^2+5}$

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{4(n+1)^2+5}}{\frac{2^n}{4n^2+5}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{2^n}\cdot 2^1(4n^2+5)}{\cancel{2^n}\cdot 4(n+1)^2+5}\right|$$

$$= \lim_{n\to\infty}\left|\frac{8n^2+10}{4n^2+8n+9}\right|$$

$$L.H \lim_{n\to\infty}\left|\frac{16n}{8n+8}\right|$$

$$L.H \lim_{n\to\infty}\left|\frac{16}{8}\right| = 2 > 1$$

$(n+1)(n+1) = n^2+2n+1$

∴ This series diverges by ratio test.

Q_{13}: Determine whether the following series

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$$

is absolutely convergent, conditionally convergent, or divergent.

Ratio Test:

Let $a_n = \frac{1}{n!}$

$$\frac{(n+1)\cdot\cancel{(n)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$a_{n+1} = \frac{1}{(n+1)!}$

$$\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0 < 1$$

∴ This series converges by ratio test.

Q_{14}: Determine whether the following series

$$\sum_{n=1}^{\infty} (6)^{-n}\, n!$$

is absolutely convergent, conditionally convergent, or divergent.

$$\frac{(n+1)\cdot\cancel{(n)}\cdot\cancel{(n-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n)}\cdot\cancel{(n-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

Ratio Test:

Let $a_n = n!$

$a_{n+1} = (n+1)!$

$$\lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty}|n+1| = \infty > 1$$

∴ This series diverges by ratio test

Q_{15}: Determine whether the following series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{5n^2+4}{2^n}$$

is absolutely convergent, conditionally convergent, or divergent.

Ratio Test:

Let $a_n = \frac{5n^2+4}{2^n}$

$a_{n+1} = \frac{5(n+1)^2+4}{2^{n+1}}$

$(n+1)(n+1) = n^2+2n+1$; $5n^2+10n+\cancel{8}\,9$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{5(n+1)^2+4}{2^{n+1}}}{\frac{5n^2+4}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{5(n+1)^2+4\cdot\cancel{2^n}}{5n^2+4\cdot\cancel{2^n}\cdot 2^1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{5n^2+10n+9}{10n^2+8}\right|$$

$$L.H \lim_{n\to\infty}\left|\frac{10n+10}{20n}\right|$$

$$L.H \lim_{n\to\infty}\left|\frac{10}{20}\right| = \frac{1}{2} < 1$$

∴ This series converges by ratio test.