This print-out should have 9 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n+3}}.$$

1. interval of cgce $= [-1, 1]$

2. interval of cgce $= (-3, 3]$

3. interval of cgce $= (-1, 1)$

4. interval of cgce $= [-1, 1)$ **correct**

5. converges only at $x = 0$

6. interval of cgce $= [-3, 3]$

Explanation:

When

$$a_n = \frac{x^n}{\sqrt{n+3}},$$

then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{\sqrt{n+4}}\frac{\sqrt{n+3}}{x^n}\right| = |x|\left(\frac{\sqrt{n+3}}{\sqrt{n+4}}\right) = |x|\sqrt{\frac{n+3}{n+4}}.$$

But

$$\lim_{n\to\infty} \frac{n+3}{n+4} = 1,$$

so

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|.$$

By the Ratio Test, therefore, the given series

(i) converges when $|x| < 1$,

(ii) diverges when $|x| > 1$.

We have still to check what happens at the endpoints $x = \pm 1$. At $x = 1$ the series becomes

$$(*) \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}.$$

Applying the Integral Test with

$$f(x) = \frac{1}{\sqrt{x+3}}$$

we see that f is continuous, positive, and decreasing on $[1, \infty)$, but the improper integral

$$I = \int_1^{\infty} f(x)\,dx$$

diverges, so the infinite series $(*)$ diverges also.

On the other hand, at $x = -1$, the series becomes

$$(\ddagger) \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}.$$

which is an alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n, \qquad a_n = f(n)$$

with

$$f(x) = \frac{1}{\sqrt{x+3}}$$

the same continuous, positive and decreasing function as before. Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{\sqrt{x+3}} = 0,$$

however, the Alternating Series Test ensures that $(\ddagger)$ converges.

Consequently, the

$$\boxed{\text{interval of convergence} = [-1, 1)}.$$

002 10.0 points

Find the radius of convergence, R, and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \sqrt{n}\,(x-4)^n.$$

1. $R = 1$, $I = [3, 5]$

2. $R = 4$, $I = (0, 4]$

3. $R = 1$, $I = (3, 5)$ **correct**

4. diverges everywhere

5. $R = 4$, $I = (-4, 4)$

Explanation:

If $a_n = \sqrt{n}\,(x-4)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}|x-4|^{n+1}}{\sqrt{n}|x-4|^n}\right| = \lim_{n\to\infty}\sqrt{1+\frac{1}{n}}\,|x-4| = |x-4|.$$

By the Ratio Test, the series converges when $|x-4| < 1$ so $-1 < x-4 < 1$. Then $3 < x < 5$. When $x = 3$, the series becomes $\sum_{n=0}^{\infty}(-1)^n\sqrt{n}$, which diverges by the Test for Divergence. When $x = 5$, the series becomes $\sum_{n=0}^{\infty}\sqrt{n}$, which also diverges by the Test for Divergence. Thus, $I = (3, 5)$.

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003 10.0 points

Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} n^3(x-4)^n\,.$$

1. interval convergence $= [3,\ 5)$

2. interval convergence $= (-5,\ -3]$

3. interval convergence $= (3,\ 5)$ **correct**

4. interval convergence $= (-\infty,\ \infty)$

5. converges only at $x = 4$

6. interval convergence $= (-5,\ -3)$

Explanation:

The given series has the form

$$\sum_{n=1} a_n(x-4)^n$$

with $a_n = n^3$. Now

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 = 1\,.$$

By the Ratio Test, therefore, the given series

(i) converges when $|x-4| < 1$, and

(ii) diverges when $|x-4| > 1$.

On the other hand, at the points $x-4 = -1$ and $x-4 = 1$ the series reduces to

$$\sum_{n=1}^{\infty}(-1)^n n^3, \qquad \sum_{n=1}^{\infty} n^3$$

respectively. But by the Divergence Test, both of these diverge. Consequently,

$$\boxed{\text{interval convergence} = (3,\ 5)}\,.$$

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004 10.0 points

Determine the radius of convergence, R, of the series

$$\sum_{n=1}^{\infty}\frac{x^n}{(n+6)!}\,.$$

1. $R = 6$

2. $R = 0$

3. $R = \infty$ **correct**

4. $R = 1$

5. $R = \dfrac{1}{6}$

Explanation:

The given series has the form

$$\sum_{n=1}^{\infty} a_n x^n$$

with

$$a_n = \frac{1}{(n+6)!}.$$

Now for this series,

(i) $R = 0$ if it converges only at $x = 0$,

(ii) $R = \infty$ if it converges for all x,

while $0 < R < \infty$,

(iii) if it converges when $|x| < R$, and

(iv) diverges when $|x| > R$.

But

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+7} = 0.$$

By the Ratio Test, therefore, the given series converges for all x. Consequently,

$$\boxed{R = \infty}.$$

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005 10.0 points

Find the interval of convergence of the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{3n+1}.$$

1. interval of cgce $= (-3,\ 1]$

2. interval of cgce $= [-1,\ 3]$

3. converges only at $x = 0$

4. interval of cgce $= (-1,\ 1]$ **correct**

5. interval of cgce $= [-1,\ 1)$

6. interval of cgce $= [-1,\ 1]$

7. interval of cgce $= (-1,\ 1)$

8. interval of cgce $= (-\infty,\ \infty)$

Explanation:
When

$$a_n = (-1)^n\frac{x^n}{3n+1},$$

then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|-\frac{x^{n+1}}{3n+4}\frac{3n+1}{x^n}\right| = |x|\left(\frac{3n+1}{3n+4}\right).$$

But

$$\lim_{n\to\infty}\frac{3n+1}{3n+4} = 1,$$

in which case

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|.$$

By the Ratio Test, therefore, the given series

(i) converges when $|x| < 1$,

(ii) diverges when $|x| > 1$.

We have still to check what happens at the endpoints $x = \pm 1$. At $x = -1$ the series becomes

$$\sum_{n=1}^{\infty}\frac{1}{3n+1}$$

which diverges by the Inegral Test. On the other hand, at $x = 1$, the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{3n+1}$$

which converges by the Alternating Series Test.

Consequently, the

$$\boxed{\text{interval of convergence} = (-1,\ 1]}.$$

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006 10.0 points

Determine the interval of convergence of the infinite series

$$\sum_{n=1}^{\infty}\frac{x^n}{4^n n^4}.$$

1. interval convergence $= [-1,\ 1)$

2. interval convergence $= [-1/4,\ 1/4)$

3. interval convergence $= [-1,\ 1]$

4. interval convergence $= [-4,\ 4)$

5. interval convergence $= [-4,\ 4]$ **correct**

6. interval convergence $= [-1/4,\ 1/4]$

7. converges only at $x = 0$

8. interval convergence $= (-\infty,\ \infty)$

Explanation:

The given series has the form

$$\sum_{n=1}^{\infty} a_n x^n$$

with

$$a_n = \frac{(-1)^n}{4^n n^4}.$$

But then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{4}\left(\frac{n}{n+1}\right)^4 = \frac{1}{4}.$$

By the Ratio Test, therefore, the given series converges when $|x| < 4$ and diverges when $|x| > 4$.

On the other hand, at the point $x = -4$ and $x = 4$, the series reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4}$$

respectively. Now

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^4}\right| = \sum_{n=1}^{\infty} \frac{1}{n^4},$$

so by the p-Series Test with $p = 4$ both series converge. Consequently, the given series has

$$\boxed{\text{interval convergence} = [-4,\ 4]}.$$

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007 10.0 points

Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}(x-5)^n.$$

1. interval convergence $= [\,3,\ 7\,]$

2. interval convergence $= (\,-2,\ 5\,)$

3. interval convergence $= [\,-2,\ 5\,)$

4. interval convergence $= [\,-2,\ 5\,]$

5. interval convergence $= [\,3,\ 7\,)$

6. interval convergence $= (\,3,\ 7\,)$ **correct**

Explanation:

Set

$$a_n = \frac{n}{2^n}(x-5)^n.$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{2^n(n+1)}{2^{n+1}n}\frac{(x-5)^{n+1}}{(x-5)^n} = \frac{(n+1)(x-5)}{2n} = \frac{x-5}{2}\left(1+\frac{1}{n}\right).$$

But

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1,$$

so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}|x-5|.$$

Thus the given series

(i) converges for $|x-5| < 2$, and

(ii) diverges for $|x-5| > 2$,

and so converges on the interval $(3,\ 7)$ and diverges outside $[3,\ 7]$.

It remains, therefore, to check convergence at the endpoints $|x-5| = 2$, *i.e.*, at $x = 3, 7$. Now at $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} (-1)^n n\,,$$

while at $x = 7$ it becomes

$$\sum_{n=1}^{\infty} n\,.$$

But by the Divergence Test, both of these series diverge. Consequently, the given series has

$$\boxed{\text{interval of convergence} \;=\; (\,3,\;7\,).}$$

keywords:

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008 10.0 points

Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n\,x^n}{\sqrt[3]{n+2}}$

1. diverges everywhere

2. $R = 4$, $I = (-4, 4]$

3. $R = \frac{1}{4}$, $I = \left[-\frac{1}{4}, \frac{1}{4}\right)$

4. $R = \frac{1}{4}$, $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$

5. $R = \frac{1}{4}$, $I = \left(-\frac{1}{4}, \frac{1}{4}\right]$ **correct**

Explanation:

$a_n = \dfrac{(-4)^n\,x^n}{\sqrt[3]{n+2}}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{4^{n+1}|x|^{n+1}}{\sqrt[3]{n+3}}\cdot\frac{\sqrt[3]{n+2}}{4^n|x|^n} = \lim_{n\to\infty} 4|x|\sqrt[3]{\frac{n+2}{n+3}} = 4|x|,$$

so by the Ratio Test, the series converges when $4|x| < 1$. Then $|x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

When $x = -\frac{1}{4}$, we get the divergent p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n+2}}$. When $x = \frac{1}{4}$, we get the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n+2}}$, which converges by the Alternating Series Test. Thus, $I = \left(-\frac{1}{4}, \frac{1}{4}\right]$.

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009 10.0 points

Determine the radius of convergence, R, of the power series

$$\sum_{n=1}^{\infty}\frac{(-4)^n}{\sqrt[3]{n}}(x+2)^n\,.$$

1. $R = \infty$

2. $R = 4$

3. $R = 0$

4. $R = \frac{1}{4}$ **correct**

5. $R = \frac{1}{2}$

6. $R = 2$

Explanation:

The given series has the form

$$\sum_{n=1}^{\infty}(-1)^n a_n (x+2)^n$$

where

$$a_n = \frac{4^n}{\sqrt[3]{n}}\,.$$

Now for this series,

(i) $R = 0$ if it converges only at $x = -2$,

(ii) $R = \infty$ if it converges for all x,

while $0 < R < \infty$

(iii) if it converges for $|x+2| < R$, and

(iv) diverges for $|x+2| > R$.

But

$$\left|\frac{a_{n+1}}{a_n}\right| = 4\Big(\frac{\sqrt[3]{n}}{\sqrt[3]{n+1}}\Big) = 4\Big(\sqrt[3]{\frac{n}{n+1}}\Big),$$

in which case

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 4.$$

By the Ratio test, therefore, the given series

(a) converges for $|x+2| < 1/4$, and

(b) diverges for $|x+2| > 1/4$.

Consequently,

$$\boxed{R = \frac{1}{4}}.$$