This print-out should have 5 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Find a power series representation for the function

$$f(z) = \frac{1}{z-4}.$$

1. $f(z) = -\sum_{n=0}^{\infty} 4^n z^n$

2. $f(z) = \sum_{n=0}^{\infty} (-1)^{n-1} 4^{n+1} z^n$

3. $f(z) = \sum_{n=0}^{\infty} \frac{1}{4^{n+1}} z^n$

4. $f(z) = -\sum_{n=0}^{\infty} \frac{1}{4^{n+1}} z^n$ **correct**

5. $f(z) = \sum_{n=0}^{\infty} (-1)^n 4^n z^n$

Explanation:

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n.$$

On the other hand,

$$\frac{1}{z-4} = -\frac{1}{4}\left(\frac{1}{1-(z/4)}\right).$$

Thus

$$f(z) = -\frac{1}{4}\sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = -\frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{4^n} z^n.$$

Consequently,

$$\boxed{f(z) = -\sum_{n=0}^{\infty} \frac{1}{4^{n+1}} z^n}$$

with $|z| < 4$.

002 10.0 points

Find a power series representation for the function

$$f(x) = \frac{1}{6+x}.$$

1. $f(x) = \sum_{n=0}^{\infty} (-1)^n 6\, x^n$

2. $f(x) = \sum_{n=0}^{\infty} \frac{1}{6^{n+1}} x^n$

3. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} x^n$ **correct**

4. $f(x) = \sum_{n=0}^{\infty} 6^{n+1} x^n$

5. $f(x) = \sum_{n=0}^{\infty} (-1)^n 6^{n+1} x^n$

Explanation:

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n.$$

On the other hand,

$$\frac{1}{6+x} = \frac{1}{6}\left(\frac{1}{1-(-x/6)}\right).$$

Thus

$$f(x) = \frac{1}{6}\sum_{n=0}^{\infty} \left(-\frac{x}{6}\right)^n = \frac{1}{6}\sum_{n=0}^{\infty} \frac{(-1)^n}{6^n} x^n.$$

Consequently,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} x^n}$$

with $|x| < 6$.

003 10.0 points

Find a power series representation for the function

$$f(x) = \frac{1}{6 - x^3}.$$

1. $f(x) = -\sum_{n=0}^{\infty} \frac{x^n}{6^{n+1}}$

2. $f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{6^{n+1}}$ **correct**

3. $f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{6^{3n}}$

4. $f(x) = \sum_{n=0}^{\infty} 6^n x^{3n}$

5. $f(x) = -\sum_{n=0}^{\infty} 6^n x^{3n}$

6. $f(x) = -\sum_{n=0}^{\infty} \frac{x^{3n}}{6^{3n}}$

Explanation:

After simplification,

$$f(x) = \frac{1}{6 - x^3} = \frac{1}{6}\left(\frac{1}{1 - (x^3/6)}\right).$$

On the other hand, we know that

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n.$$

Replacing t with $x^3/6$, we thus obtain

$$\boxed{f(x) = \frac{1}{6}\sum_{n=0}^{\infty} \frac{x^{3n}}{6^n} = \sum_{n=0}^{\infty} \frac{x^{3n}}{6^{n+1}}}.$$

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keywords:

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004 10.0 points

Find a power series representation for

$$\frac{4+3x}{1+x}.$$

Hint: separate then use the series for $\dfrac{1}{1+x}$.

1. $\frac{4+3x}{1+x} = 4 + \sum_{k=0}^{\infty} (-1)^k x^k$

2. $\frac{4+3x}{1+x} = \sum_{k=1}^{\infty} (-1)^k x^k$

3. $\frac{4+3x}{1+x} = 7\sum_{k=1}^{\infty} x^k$

4. $\frac{4+3x}{1+x} = 4 + 7\sum_{k=1}^{\infty} x^k$

5. $\frac{4+3x}{1+x} = 4 + 7\sum_{k=0}^{\infty} x^k$

6. $\frac{4+3x}{1+x} = 4 + \sum_{k=1}^{\infty} (-1)^k x^k$ **correct**

Explanation:

Using the hint we get

$$\frac{4+3x}{1+x} = \frac{4}{1+x} + \frac{3x}{1+x},$$

and

$$\frac{1}{1+x} = 1 - x + x^2 + \ldots = \sum_{k=0}^{\infty} (-1)^k x^k.$$

Thus

$$\frac{4+3x}{1+x} = 4\sum_{k=0}^{\infty} (-1)^k x^k + 3x\sum_{k=0}^{\infty} (-1)^k x^k.$$

But

$$4\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-1)^k 4\, x^k,$$

while

$$3x\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-1)^k 3\, x^{k+1}.$$

To combine the infinite sums we need to express the last one as a sum of powers of x^k:

$$\sum_{k=0}^{\infty}(-1)^k 3\,x^{k+1} = 3x - 3x^2 + 3x^3 - \ldots$$

$$= -\sum_{k=1}^{\infty}(-1)^k 3\,x^k\,.$$

Since the last sum now goes from $k = 1$ to $k = \infty$, we next write:

$$\sum_{k=0}^{\infty}(-1)^k 4x^k = 4 + \sum_{k=1}^{\infty}(-1)^k 4\,x^k\,,$$

for then we can add the two series:

$$\sum_{k=0}^{\infty}(-1)^k 4\,x^k + \sum_{k=0}^{\infty}(-1)^k 3\,x^{k+1}$$

$$= 4 + \sum_{k=1}^{\infty}(-1)^k 4\,x^k + \Big(-\sum_{k=1}^{\infty}(-1)^k 3\,x^k\Big)\,.$$

Consequently,

$$\boxed{\frac{4+3x}{1+x} = 4 + \sum_{k=1}^{\infty}(-1)^k x^k}\,.$$

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005 10.0 points

Evaluate the integral

$$f(t) = \int_0^t \frac{s}{1-s^4}\,ds\,.$$

as a power series.

1. $f(t) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n t^{4n+2}}{4n+2}$

2. $f(t) = \displaystyle\sum_{n=0}^{\infty}\frac{t^{4n+2}}{4n+2}$ **correct**

3. $f(t) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n t^{4n}}{4n}$

4. $f(t) = \displaystyle\sum_{n=0}^{\infty}\frac{t^{4n}}{4n}$

5. $f(t) = \displaystyle\sum_{n=4}^{\infty}\frac{t^{4n}}{4n+2}$

Explanation:

By the geometric series representation,

$$\frac{1}{1-s} = \sum_{n=0}^{\infty} s^n\,,$$

and so

$$\frac{s}{1-s^4} = \sum_{n=0}^{\infty} s^{4n+1}\,.$$

But then

$$f(t) = \int_0^t \Big(\sum_{n=0}^{\infty} s^{4n+1}\Big)\,ds$$

$$= \sum_{n=0}^{\infty}\Big(\int_0^t s^{4n+1}\,ds\Big)\,.$$

Consequently,

$$\boxed{f(t) = \sum_{n=0}^{\infty}\frac{t^{4n+2}}{4n+2}}\,.$$