This print-out should have 4 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Find the Taylor series centered at $x = 0$ for

$$f(x) = \cos(4x).$$

1. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(2n)!}\, x^{2n}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{(2n)!}\, x^{2n}$ **correct**

3. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{(2n)!}\, x^{n}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{n!}\, x^{n}$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{n!}\, x^{2n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(2n)!}\, x^{n}$

Explanation:

The Taylor series centered at $x = 0$ for any f is

$$f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)x^n.$$

But when $f(x) = \cos(4x)$,

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos(4x)$	1
1	$-4\sin(4x)$	0
2	$-16\cos(4x)$	-16
3	$64\sin(4x)$	0
4	$256\cos(4x)$	256
.	.	.
.	.	.

in other words,

$$f'(0) = f'''(0) = f^{(5)}(0) = \ldots = 0,$$

while

$$f(0) = 1, \quad f''(0) = -4^2, \quad f^{(4)}(0) = 4^4.$$

Thus in general,

$$f^{(2n+1)}(0) = 0, \quad f^{(2n)}(0) = (-1)^n 4^{2n}.$$

Consequently, the Taylor series of $\cos(4x)$ centered at $x = 0$ is

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{(2n)!}\, x^{2n}}.$$

002 10.0 points

Find the Taylor series representation for f centered at $x = 1$ when

$$f(x) = 4 + 5x - 3x^2.$$

1. $f(x) = 6 - (x-1) - 3(x-1)^2$ **correct**

2. $f(x) = 6 - (x-1) - 6(x-1)^2$

3. $f(x) = 4 - (x-1) + 6(x-1)^2$

4. $f(x) = 6 + 5(x-1) + 3(x-1)^2$

5. $f(x) = 4 + 5(x-1) - 6(x-1)^2$

6. $f(x) = 4 + 5(x-1) - 3(x-1)^2$

Explanation:

For a function f the Taylor series representation centered at $x = 1$ is given by

$$f(x) = \sum_{n=0} \frac{1}{n!} f^{(n)}(1)(x-1)^n.$$

Since f is a polynomial of degree 2, however, $f^{(n)} = 0$ for all $n \geq 3$, so we have only to calculate derivatives of f up to order 2:

$$f'(x) = 5 - 6x, \qquad f''(x) = -6.$$

Thus

$$f(1) = 6, \quad f'(1) = -1, \quad f''(1) = -6.$$

Consequently,

$$\boxed{f(x) = 6 - (x-1) - 3(x-1)^2}.$$

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003 10.0 points

Find the coefficient of x^4 in the Taylor series expansion centered at the origin for the function

$$f(x) = 4\ln(5 - 8x^2).$$

1. coefficient of $x^4 = -\dfrac{128}{25}$ **correct**

2. coefficient of $x^4 = -\dfrac{8192}{25}$

3. coefficient of $x^4 = -\dfrac{128}{625}$

4. coefficient of $x^4 = \dfrac{32}{625}$

5. coefficient of $x^4 = \dfrac{128}{25}$

Explanation:

The Taylor series expansion of

$$F(x) = 4\ln(5 - 8x^2)$$

centered at the origin is given by

$$F(x) = 4\Big(\ln(5) - \frac{8x}{5} - \frac{8^2x^2}{2\cdot 5^2} - \frac{8^3x^3}{3\cdot 5^3} - \dots\Big)$$

and this holds on the interval $\left(-\dfrac{5}{8}, \dfrac{5}{8}\right)$. Thus the Taylor series expansion centered at the origin for $f(x)$ is

$$f(x) = 4\Big(\ln(5) - \frac{8x^2}{5} - \frac{32}{25}x^4 - \frac{512}{375}x^6 - \dots\Big);$$

it has interval of convergence

$$\left(-\sqrt{\frac{5}{8}}, \sqrt{\frac{5}{8}}\right).$$

By inspection, therefore,

$$\boxed{\text{coefficient of } x^4 = -\frac{128}{25}}.$$

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004 10.0 points

Find a power series representation centered at the origin for the function

$$f(x) = (6+x)^{-3}.$$

1. $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{6^{n+2}}\, x^n$

2. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{6^{n+3}}\, x^n$

3. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{6^{n+2}}\, x^n$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2\cdot 6^{n+3}}\, x^n$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2\cdot 6^{n+2}}\, x^n$

6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2\cdot 6^{n+3}}\, x^n$ **correct**

Explanation:

Since

$$\frac{1}{(1+x)^3} = -\frac{1}{2}\frac{d}{dx}\left(\frac{1}{(1+x)^2}\right) = \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{1+x}\right)$$

we can begin with geometric series representation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

and differentiate twice. But then

$$\frac{d^2}{dx^2}\left(\frac{1}{1+x}\right)$$

$$= 2 - 2\cdot 3x + 3\cdot 4x^2 - 4\cdot 5x^3 + \ldots$$

$$= \sum_{n=2}^{\infty} (-1)^n n(n-1)\, x^{n-2}\,,$$

and so

$$\frac{1}{(1+x)^3} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} x^n\,.$$

Consequently, if we write

$$\frac{1}{(6+x)^3} = \frac{1}{6^3}\frac{1}{(1+(x/6))^3}\,,$$

then

$$\boxed{\frac{1}{(6+x)^3} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2\cdot 6^{n+3}}\, x^n}\,.$$

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keywords: geometric series, differentiate, derivative, power series,