This print-out should have 7 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Find the degree 2 Taylor polynomial of f centered at $x = 2$ when

$$f(x) = 5x \ln x .$$

1. $10 + 5\ln 2(x-2) + \frac{5}{2}(x-2)^2$

2. $10 + 5(\ln 2 + 1)(x-2) + \frac{5}{4}(x-2)^2$

3. $10 + 2\ln 5(x-2) + \frac{5}{4}(x-2)^2$

4. $10\ln 2 + 5(\ln 2 + 1)(x-2) + \frac{5}{2}(x-2)^2$

5. $10\ln 2 + 5\ln 2(x-2) + \frac{5}{4}(x-2)^2$

6. $10\ln 2 + 5(\ln 2 + 1)(x-2) + \frac{5}{4}(x-2)^2$ **correct**

Explanation:

The degree 2 Taylor polynomial of f centered at $x = 2$ is given by

$$T_2(x) = f(2) + f'(2)(x-2) + \frac{1}{2!}f''(2)(x-2)^2 .$$

When $f(x) = 5x \ln x$, therefore,

$$f'(x) = 5\ln x + 5, \qquad f''(x) = \frac{5}{x} .$$

But when $f(2) = 10\ln 2$,

$$f'(2) = 5(\ln 2 + 1) , \qquad f''(2) = \frac{5}{2} .$$

Consequently, the degree 2 Taylor polynomial centered at $x = 2$ of f is

$$\boxed{10\ln 2 + 5(\ln 2 + 1)(x-2) + \frac{5}{4}(x-2)^2} .$$

002 10.0 points

Determine the degree three Taylor polynomial centered at $x = 1$ for f when

$$f(x) = e^{2-3x} .$$

1. $T_3 = e^5\Big(1 + 3x - \frac{9}{2}x^2 + \frac{9}{2}x^3\Big)$

2. $T_3 = e^{-1}\Big(1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3\Big)$

3. $T_3 = 1 - 3(x-1) + \frac{9}{2}(x-1)^2 - \frac{9}{2}(x-1)^3$

4. $T_3 = e^{-1}\Big(1 - 3(x-1) + \frac{9}{2}(x-1)^2 - \frac{9}{2}(x-1)^3\Big)$ **correct**

5. $T_3 = e^5\Big(1 + 3(x-1) - \frac{9}{2}(x-1)^2 + \frac{9}{2}(x-1)^3\Big)$

Explanation:

The degree three Taylor polynomial centered at $x = 1$ for a function f is defined by

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \frac{1}{3!}f'''(1)(x-1)^3.$$

When $f(x) = e^{2-3x}$ we use the Chain Rule repeatedly to compute the derivatives of f:

$$f'(x) = -3e^{2-3x}, \quad f''(x) = 3^2e^{2-3x},$$

and

$$f'''(x) = -3^3 e^{2-3x}.$$

Thus

$$f(1) = e^{-1}, \qquad f'(1) = -3e^{-1},$$

$$f''(1) = 3^2 e^{-1}, \qquad f'''(1) = -3^3 e^{-1}.$$

Consequently,

$$\boxed{T_3 = e^{-1}\Big(1 - 3(x-1) + \frac{9}{2}(x-1)^2 - \frac{9}{2}(x-1)^3\Big).}$$

003 10.0 points

Find the degree three Taylor polynomial T_3 centered at $x = 0$ for f when

$$f(x) = \ln(2 - 3x).$$

1. $T_3(x) = \ln 2 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{9}{16}x^3$

2. $T_3(x) = \frac{3}{2}x + \frac{9}{8}x^2 + \frac{9}{8}x^3$

3. $T_3(x) = \frac{3}{2}x - \frac{9}{8}x^2 - \frac{9}{8}x^3$

4. $T_3(x) = \ln 2 - \frac{3}{2}x + \frac{9}{8}x^2 - \frac{9}{8}x^3$

5. $T_3(x) = \frac{3}{2}x - \frac{9}{8}x^2 + \frac{9}{8}x^3$

6. $T_3(x) = \ln 2 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{9}{8}x^3$ **correct**

Explanation:

The degree three Taylor polynomial centered at $x = 0$ for a function f is defined by

$$p_3(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3.$$

We use the Chain Rule repeatedly to compute the derivatives of f:

$$f'(x) = -\frac{3}{2-3x}, \quad f''(x) = -\frac{9}{(2-3x)^2},$$

and

$$f'''(x) = -\frac{54}{(2-3x)^3}.$$

Thus

$$f(0) = \ln 2, \quad f'(0) = -\frac{3}{2},$$

$$\frac{1}{2!}f''(0) = -\frac{9}{8}, \quad \frac{1}{3!}f'''(0) = -\frac{9}{8},$$

and so

$$\boxed{T_3(x) = \ln 2 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{9}{8}x^3}.$$

004 10.0 points

Find the Taylor series centered at the origin for the function

$$f(x) = x\cos(6x).$$

1. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1}$

2. $f(x) = \sum_{n=0}^{\infty} \frac{6^n}{n!} x^{n+1}$

3. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n}}{(2n)!} x^{2n+1}$ **correct**

4. $f(x) = \sum_{n=0}^{\infty} \frac{6^{2n}}{(2n)!} x^{2n+1}$

5. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 6^n}{n!} x^{n+1}$

Explanation:

The Taylor series centered at the origin for $\cos(x)$ is

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

But then

$$x\cos(6x) = x\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (6x)^{2n}.$$

Consequently, the Taylor series representation for f centered at the origin is

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n}}{(2n)!} x^{2n+1}}.$$

005 10.0 points

Use the degree 2 Taylor polynomial centered at the origin for f to estimate the integral

$$I = \int_0^1 f(x)\,dx$$

when

$$f(x) = e^{-x^2/2}.$$

1. $I \approx \frac{5}{6}$ **correct**
2. $I \approx \frac{1}{3}$
3. $I \approx \frac{1}{2}$
4. $I \approx 1$
5. $I \approx \frac{2}{3}$

Explanation:

When $f(x) = e^{-x^2/2}$, we see that

$$f'(x) = -xe^{-x^2/2},$$

while

$$f''(x) = -e^{-x^2/2} + x^2 e^{-x^2/2}.$$

In this case,

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1.$$

Thus the degree 2 Taylor polynomial for f centered at the origin is

$$T_2(x) = 1 - \frac{1}{2}x^2.$$

But then

$$I \approx \int_0^1 T_2(x)\,dx = \int_0^1 \left(1 - \frac{1}{2}x^2\right) dx.$$

Consequently,

$$\boxed{I \approx \left[x - \frac{1}{6}x^3\right]_0^1 = \frac{5}{6}}.$$

006 10.0 points

Use the degree 2 Taylor polynomial centered at the origin for f to estimate the integral

$$I = \int_0^1 f(x)\,dx$$

when

$$f(x) = \sqrt{1+x^2}.$$

1. $I \approx 1$
2. $I \approx \frac{2}{3}$
3. $I \approx \frac{7}{6}$ **correct**
4. $I \approx \frac{4}{3}$
5. $I \approx \frac{5}{6}$

Explanation:

When

$$f(x) = \sqrt{1+x^2} = (1+x^2)^{1/2},$$

we see that

$$f'(x) = x(1+x^2)^{-1/2},$$

while

$$f''(x) = (1+x^2)^{-1/2} - x^2(1+x^2)^{-3/2}.$$

In this case,

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = 1.$$

Thus the degree 2 Taylor polynomial for f centered at the origin is

$$T_2(x) = 1 + \frac{1}{2}x^2.$$

But then

$$I \approx \int_0^1 T_2(x)\,dx = \int_0^1 \left(1 + \frac{1}{2}x^2\right) dx.$$

Consequently,

$$\boxed{I \approx \left[x + \frac{1}{6}x^3\right]_0^1 = \frac{7}{6}}.$$

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007 10.0 points

Use the Taylor series for e^{-x^2} to evaluate the integral

$$I = \int_0^3 2\,e^{-x^2}\,dx.$$

1. $I = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} 2\cdot 3^{2k+1}$ **correct**

2. $I = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} 2\cdot 3^{2k}$

3. $I = \sum_{k=0}^{n} \frac{1}{k!(2k+1)} 2\cdot 3^{2k+1}$

4. $I = \sum_{k=0}^{\infty} \frac{1}{k!} 2\cdot 3^{2k}$

5. $I = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} 2\cdot 3^{2k+1}$

Explanation:

The Taylor series for e^x is given by

$$e^x = 1 + x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n + \ldots$$

and its interval of convergence is $(-\infty, \infty)$. Thus we can substitute $x \to -x^2$ for all values of x, showing that

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}x^{2k}$$

everywhere on $(-\infty, \infty)$. Thus

$$I = \int_0^3 2\Big(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}x^{2k}\Big)\,dx.$$

But we can change the order of summation and integration on the interval of convergence, so

$$I = 2\sum_{k=0}^{\infty}\Big(\int_0^3 \frac{(-1)^k}{k!}x^{2k}\Big)\,dx$$

$$= 2\sum_{k=0}^{\infty}\Big[\frac{(-1)^k}{k!(2k+1)}x^{2k+1}\Big]_0^3.$$

Consequently,

$$\boxed{I = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} 2\cdot 3^{2k+1}}.$$