Question #1

$I = \int \frac{x^2}{(4-x^2)^{3/2}}\,dx \longrightarrow 4 - (2\sin\theta)^2$

$x = 2\sin\theta$ $\quad 4(\cancel{4}-\cancel{4}\sin^2\theta)$

$dx = 2\cos\theta\,d\theta$ $\quad 4(1-\sin^2\theta)$

$4\cos^2\theta$

$(4\cos^2\theta)^{3/2}$ $\quad \sqrt{4^3}$

$8\cos^3\theta$

$\frac{x}{2} = \frac{2\sin\theta}{2}$

$\arcsin\left(\frac{x}{2}\right) = \sin\theta$

$\arcsin\left(\frac{x}{2}\right) = \theta$

$\sin\theta = \frac{x}{2}$

$\int \frac{(2\sin\theta)^2}{8\cos^3\theta} = \int \frac{4\sin^2\theta}{8\cos^3\theta}\cdot 2\cos\theta\,d\theta$

$\frac{\cancel{4}\sin^2\theta}{\cancel{4}\cos^2\theta}$

$\int \frac{\sin^2\theta}{\cos^2\theta} = \int \tan^2\theta\,d\theta$

$\int \sec^2\theta - 1 \longrightarrow \int \sec^2\theta\,d\theta - \int 1\,d\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{x}{2}}{\quad}$

$1 - \sin^2\theta = \cos^2\theta$

$1 - \left(\frac{x}{2}\right)^2$

$1 - \frac{x^2}{4} = \cos^2\theta$

$\sqrt{\frac{4-x^2}{4}} = \sqrt{\cos^2\theta}$

$\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}$

$I = \tan\theta - \theta + C$

$I = \frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{x}{2}\right) + C$

$\frac{x}{\sqrt{4-x^2}} = \frac{x}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{4-x^2}}$

Question #2

$I = \int_0^1 \frac{3x^2}{(2-x^2)^{3/2}}$

$\int_0^1 \frac{3x^2}{\left(\sqrt{2-x^2}\right)^3}$

$\frac{\sqrt{2}\sin\theta}{\sqrt{2}} = \frac{x}{\sqrt{2}}$

$\arcsin\left(\frac{x}{\sqrt{2}}\right) = \theta$

$\sqrt{2}\sin\theta = x$

$d\theta\,\sqrt{2}\cos\theta = dx$

$x = 1 \to \theta_1 = \frac{\pi}{4}$

$x = 0 \quad \theta_1 = 0$

$2 - (\sqrt{2}\sin\theta)^2 = 2 - 2\sin^2\theta$

$= 2(1-\sin^2\theta)$

$2 - x^2 = 2\cos^2\theta$

$(2\cos^2\theta)^{3/2}$

$2^{3/2}\cos^3\theta$

$2\sqrt{2}\cos^3\theta$

$\frac{3(2\sin^2\theta)}{2\sqrt{2}\cos^3\theta}\cdot\sqrt{2}\cos\theta\,d\theta$

$\frac{6\sin^2\theta}{2\cancel{\sqrt{2}}\cancel{\cos\theta}}\cdot\cancel{\sqrt{2}\cos\theta}\,d\theta = \frac{6\sin^2\theta}{2\cos^2\theta}$

$3\int \tan^2\theta\,d\theta$

$3\int \sec^2\theta - 1\,d\theta$

$\int \sec^2\theta - \int 1\,d\theta$

$3(\tan\theta - \theta)\Big]_0^{\frac{\pi}{4}}$

$3\left(\left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - (\tan(0) - 0)\right)$

$\left(1 - \frac{\pi}{4}\right) - (0)$

$3\left(1 - \frac{\pi}{4}\right)$

Question #3

Evaluate the integral

$I = \int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx$

$\int_0^2 \frac{1}{\sqrt{4^2-x^2}}\,dx$

$x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$\frac{x}{4} = \frac{4\sin\theta}{4}$

$\arcsin\left(\frac{x}{4}\right) = \theta$

$x_1 \to \theta_1 : 2 \to$

$x_2 \to \theta_2 : 0 \to$

$16 - (4\sin\theta)^2$

$\sqrt{16 - 16\sin^2\theta}$

$16(1-\sin^2\theta)$

$\sqrt{16\cos^2\theta}$

$\int \frac{4\cos\theta}{4\cos\theta}\,d\theta$

$I = \int d\theta$

$I = \theta\Big]_0^{\pi/6}$

$I = \frac{\pi}{6} - 0 = \boxed{\frac{1}{6}\pi}$

Question #4

Evaluate the integral

$I = \int_{\sqrt{2}}^{2} \frac{6}{x\sqrt{x^2-1}}\,dx$

$x = \sqrt{1}\sec\theta$

$dx = \sqrt{1}\sec\theta\tan\theta\,d\theta$

$(\sqrt{1}\sec\theta)^2 - 1$

$\sec^2\theta - 1 = \tan^2\theta$

$\sqrt{\tan^2\theta}$

$\frac{6}{\cancel{\sqrt{1}\sec\theta\tan\theta}}\cdot\cancel{\sqrt{1}\sec\theta\tan\theta}\,d\theta$

$\sec^{-1}\left(\frac{x}{\sqrt{1}}\right)^{x=2} = \frac{\pi}{3}$

$\sec^{-1}\left(\frac{x}{\sqrt{1}}\right)^{x=\sqrt{2}} = \frac{\pi}{4}$

$\int 6\,d\theta$

$6\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$(4)\ \frac{\pi}{3} - \frac{\pi}{4}\ (3)$

$\frac{4\pi - 3\pi}{12} = \left(\frac{\pi}{12}\right)6$

$I = \frac{1}{2}\pi$

Question #5

Evaluate the integral

$I = \int_0^2 \frac{x^2+5}{4+x^2}\,dx$ $\quad x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\frac{4\tan^2\theta + 5}{4 + 4\tan^2\theta}\,2\sec^2\theta\,d\theta$

$\frac{4\tan^2\theta + 5}{4(\cancel{1+\tan^2\theta})}\,2\sec^2\theta\,d\theta$

$4(\sec^2\theta)$

$\frac{4\tan^2\theta + 5}{2}$

$\frac{1}{2}\int_0^{\pi/4} 4\tan^2\theta + 5$

$\frac{1}{2}\int_0^{\pi/4} 4(\sec^2 x - 1) + 5$

$\frac{1}{2}\left(4\int_0^{\pi/4}\sec^2 x - 1 + \int_0^{\pi/4} 5\,d\theta\right)$

$4\left(\tan\theta - \theta\Big]_0^{\pi/4}\right) + \frac{5\pi}{4}$

$4\left(1 - \frac{\pi}{4}\right)$ $\quad 4 - \frac{4\pi}{4} + \frac{5\pi}{4} \longrightarrow \left(4 + \frac{\pi}{4}\right)\frac{1}{2} = \boxed{2 + \frac{1}{8}\pi}$

$2 - \frac{\pi}{4} + \frac{5\pi}{4}$

Question #6

Evaluate the integral (To which of the following does the integral reduce after an appropriate trig substitution.)

$I = \int \frac{x^5}{\sqrt{1-x^2}}$ $\quad x = \sqrt{1}\sin\theta$

$dx = \sqrt{1}\cos\theta$

$1 - (\sqrt{1}\sin\theta)^2$

$1 - \sin^2\theta = \cos^2\theta$

$\sqrt{\cos^2\theta} = \cos\theta$

$\int \frac{\sin^5\theta}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\,d\theta$

$\int \sin^5\theta\,d\theta$ ✓

Question #7

To which one of the following does the integral

$I = \int \frac{x^2}{\sqrt{x^2+1}}\,dx$

reduce after an appropriate trig substitution?

$\int \frac{x^2}{\sqrt{x^2+1}}$ $\quad x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\frac{\tan^2\theta}{\sqrt{\tan^2\theta + 1}}\cdot\sec^2\theta\,d\theta$

$\frac{\tan^2\theta}{\sqrt{\sec^2\theta}}\cdot\sec^2\theta\,d\theta \to \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\,d\theta$

$\tan^2\theta\sec\theta\,d\theta$

$\frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{1}{\cos\theta}\,d\theta$

$\frac{\sin^2\theta}{\cos^3\theta} \to \int \sin^2\theta\sec^3\theta\,d\theta$

Question #8

Evaluate the integral

$I = \int_0^{1/4} \frac{3}{\sqrt{1-4x^2}}\,dx$

$\sqrt{4\left(\frac{1}{4} - x^2\right)}$

$2\sqrt{\frac{1}{4} - x^2}$

$\frac{3}{2}\int \frac{1}{\sqrt{\frac{1}{4} - x^2}}\,dx$ $\quad x = \frac{1}{2}\sin\theta$

$dx = \frac{1}{2}\cos\theta\,d\theta$

$\sqrt{a^2 - x^2} \to x = a\sin\theta$

$\sqrt{a^2 + x^2} \to x = a\tan\theta$

$\sqrt{x^2 - a^2} \to x = a\sec\theta$

$\arcsin(2x) = \theta$

$\frac{3}{2}\int \frac{1}{\sqrt{\frac{1}{4} - \left(\frac{1}{2}\sin\theta\right)^2}}$

$\frac{1}{\sqrt{\frac{1}{4} - \frac{1}{4}\sin^2\theta}}$

$\left(\frac{1}{4}\right)(1 - \sin^2\theta) \to \cos^2\theta$

$\frac{3}{2}\int_0^{\pi/6} \frac{1}{\frac{1}{2}\sqrt{1-\sin^2\theta}}\cdot\frac{1}{2}\cos\theta\,d\theta$

$\cos$

$3\int_0^{\pi/6} d\theta$ $\quad \frac{3}{2}\arcsin\left(\frac{2}{4}\right) = \frac{3}{2}\cdot\frac{\pi}{6}$ $\quad \frac{3\pi}{12} = \frac{1}{4}\pi$

$\frac{3}{2}\arcsin(0) = 0$

Question #9

Evaluate the integral

$I = \int_0^1 \frac{1}{\sqrt{4 - 3x^2}}\,dx$

$\frac{1}{\sqrt{3\left(\frac{4}{3} - x^2\right)}}$ $\quad x = \sqrt{\frac{4}{3}}\sin\theta$

$dx = \sqrt{\frac{4}{3}}\cos\theta\,d\theta$

$\frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\frac{4}{3} - \frac{4}{3}\sin^2\theta}}$ $\quad \frac{1}{\sqrt{3}}\arcsin\left(\frac{1}{\sqrt{\frac{4}{3}}}x\right)\Big|_0^1$

$\frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\frac{4}{3}(1-\sin^2\theta)}}\cdot\cancel{\sqrt{\frac{4}{3}}\cos\theta}\,d\theta$

$\cancel{\sqrt{\frac{4}{3}}\cos\theta}$

$\frac{1}{\sqrt{3}}\int d\theta$

$\frac{1}{\sqrt{3}}\theta$

Question #10

Evaluate the integral

$I = \int_0^1 \frac{3}{\sqrt{x^2+1}}$ $\quad x = \tan\theta$

$dx = \sec^2\theta$

$\tan^{-1}(x) = \theta$

$\frac{3}{\cancel{\sec\theta}}\cdot\sec^{\cancel{2}}\theta\,d\theta$ $\quad \sqrt{\tan^2\theta + 1}$

$\sqrt{\sec^2\theta} = \sec\theta$

$\int_0^{\frac{\pi}{4}} 3\sec\theta\,d\theta$

$3\int_0^{\frac{\pi}{4}} \sec\theta\,d\theta$

$3\left(\ln|\sec\theta + \tan\theta|\Big]_0^{\frac{\pi}{4}}\right)$

$\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \cancel{\ln|\sec 0 + \tan 0|} \to 0$

$3\left(\ln|\sqrt{2} + 1|\right)$

$3\ln(\sqrt{2}+1)$

Question #11 *

Evaluate the integral

$I = \int_0^2 \left(6 - \sqrt{4-x^2}\right)dx$ $\quad x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$\arcsin\left(\frac{x}{2}\right) = \theta$

$\int_0^2 6\,dx - \int_0^2 \sqrt{4-x^2}\,dx$

$12 -$

$\sqrt{4 - (2\sin\theta)^2}$

$\sqrt{4 - 4\sin^2\theta}$

$\sqrt{4(1-\sin^2\theta)}$

$2\cdot 2\sqrt{1-\sin^2\theta}$ $\quad \cos\theta\,d\theta$

$\sqrt{\cos^2\theta}$

$4\cos\theta\,\cos\theta\,d\theta$

$4\cos^2\theta$

$4\int \cos^2\theta\,d\theta$

$4\int \frac{1+\cos 2\theta}{2}\,d\theta$

$2\int_0^{\pi/2} 1 + \cos 2\theta\,d\theta$

$\left(\frac{1}{2}\sin\left(2\left(\frac{\pi}{2}\right)\right) - \frac{1}{2}\sin(0)\right)$

$2\left(\frac{\pi}{2} + 0\right)$

$12 - \pi = I$

Question #12

Evaluate the integral

$I = \int_0^1 \frac{x^2}{1+x^2}\,dx$ $\quad x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\int_0^{\pi/4} \frac{\tan^2\theta}{1+\tan^2\theta}\sec^2\theta\,d\theta$

$\frac{\tan^2\theta}{\cancel{\sec^2\theta}}\cancel{\sec^2\theta}\,d\theta$

$\int_0^{\pi/4} \tan^2\theta\,d\theta$

$\sec^2\theta - 1\,d\theta$

$\int_0^{\pi/4}\sec^2\theta - \int_0^{\pi/4} 1\,d\theta$

$\tan\theta\Big]_0^{\pi/4} - \theta\Big]_0^{\pi/4}$

$\left(\tan\left(\frac{\pi}{4}\right) - \tan(0)\right) - \frac{\pi}{4}$

$1 - \frac{\pi}{4}$

$\frac{4}{4} - \frac{\pi}{4}$

$\frac{1}{4}(4 - \pi)$

2)

; -4