This print-out should have 12 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Determine the integral

$$I = \int \frac{x^2}{(4-x^2)^{3/2}}\,dx\,.$$

1. $I = \dfrac{x}{\sqrt{4-x^2}} + \sin^{-1}\Big(\dfrac{x}{2}\Big) + C$

2. $I = \dfrac{2x}{\sqrt{4-x^2}} - \sin^{-1}\Big(\dfrac{x^2}{4}\Big) + C$

3. $I = \dfrac{2x^2}{\sqrt{4-x^2}} + \sin^{-1}\Big(\dfrac{x^2}{4}\Big) + C$

4. $I = \dfrac{x^2}{\sqrt{4-x^2}} + \sin^{-1}\Big(\dfrac{x^2}{2}\Big) + C$

5. $I = \dfrac{2x}{\sqrt{4-x^2}} - \sin^{-1}\Big(\dfrac{x}{4}\Big) + C$

6. $I = \dfrac{x}{\sqrt{4-x^2}} - \sin^{-1}\Big(\dfrac{x}{2}\Big) + C$ **correct**

Explanation:

Let $x = 2\sin\theta$. Then

$$dx = 2\cos\theta\,d\theta\,, \quad 4-x^2 = 4\cos^2\theta\,.$$

In this case,

$$I = \int \frac{4\cdot 2\sin^2\theta\,\cos\theta}{2^3\cos^3\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta\,.$$

Now

$$\tan^2\theta = \sec^2\theta - 1\,, \quad \frac{d}{d\theta}\tan\theta = \sec^2\theta\,,$$

and so

$$I = \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C\,.$$

Consequently,

$$\boxed{I = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\Big(\frac{x}{2}\Big) + C}$$

with C an arbitrary constant.

002 10.0 points

Evaluate the integral

$$I = \int_0^1 \frac{3x^2}{(2-x^2)^{3/2}}\,dx\,.$$

1. $I = \sqrt{3} - \dfrac{\pi}{3}$

2. $I = 3\Big(\sqrt{2} - \dfrac{\pi}{4}\Big)$

3. $I = 3\Big(1 + \dfrac{\pi}{4}\Big)$

4. $I = \sqrt{3} + \dfrac{\pi}{3}$

5. $I = \sqrt{2} + \dfrac{\pi}{3}$

6. $I = 3\Big(1 - \dfrac{\pi}{4}\Big)$ **correct**

Explanation:

Let $x = \sqrt{2}\sin\theta$. Then

$$dx = \sqrt{2}\,\cos\theta\,d\theta\,, \quad 2-x^2 = 2\cos^2\theta\,,$$

while

$$x = 0 \implies \theta = 0\,,$$

$$x = 1 \implies \theta = \frac{\pi}{4}\,.$$

In this case,

$$I = 3\int_0^{\pi/4} \frac{2\sqrt{2}\,\sin^2\theta\,\cos\theta}{2\sqrt{2}\,\cos^3\theta}\,d\theta$$

$$= 3\int_0^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = 3\int_0^{\pi/4} \tan^2\theta\,d\theta\,.$$

Now

$$\tan^2\theta = \sec^2\theta - 1\,, \quad \frac{d}{d\theta}\tan\theta = \sec^2\theta\,,$$

and so

$$I = 3\int_0^{\pi/4} (\sec^2\theta - 1)\,d\theta = 3\Big[\tan\theta - \theta\Big]_0^{\pi/4}\,.$$

Consequently,

$$\boxed{I = 3\Big(1 - \frac{\pi}{4}\Big)}.$$

003 10.0 points

Evaluate the integral

$$I = \int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx\,.$$

1. $I = \dfrac{1}{3}$

2. $I = \dfrac{1}{6}\pi$ **correct**

3. $I = \dfrac{1}{6}$

4. $I = \dfrac{1}{4}$

5. $I = \dfrac{1}{4}\pi$

6. $I = \dfrac{1}{3}\pi$

Explanation:
Set $x = 4\sin u$; then

$$dx = 4\,\cos u\,du$$

and

$$16 - x^2 = 16(1-\sin^2 u) = 8\cos^2 u\,,$$

while

$$x = 0 \implies u = 0,$$

$$x = 2 \implies u = \frac{\pi}{6}\,.$$

In this case

$$I = \int_0^{\pi/6} \frac{\cos u}{\cos u}\,du = \int_0^{\pi/6}\,du\,.$$

Consequently

$$\boxed{I = \frac{1}{6}\pi}.$$

004 10.0 points

Evaluate the integral

$$I = \int_{\sqrt{2}}^2 \frac{6}{x\sqrt{x^2-1}}\,dx\,.$$

1. $I = \dfrac{3}{4}$

2. $I = \dfrac{1}{2}$

3. $I = 1$

4. $I = \dfrac{3}{4}\pi$

5. $I = \pi$

6. $I = \dfrac{1}{2}\pi$ **correct**

Explanation:
Set $x = \sec u$. Then

$$dx = \sec u \tan u\,du\,, \quad x^2 - 1 = \tan^2 u\,,$$

while

$$x = \sqrt{2} \implies u = \frac{\pi}{4}\,,$$

$$x = 2 \implies u = \frac{\pi}{3}\,.$$

In this case,

$$I = 6\int_{\pi/4}^{\pi/3} \frac{\sec u\,\tan u}{\sec u\,\tan u}\,du = \int_{\pi/4}^{\pi/3} 6\,du\,.$$

Consequently,

$$\boxed{I = 6\Big(\frac{\pi}{3} - \frac{\pi}{4}\Big) = \frac{1}{2}\pi}.$$

005 10.0 points

Evaluate the integral

$$I = \int_0^2 \frac{x^2+5}{4+x^2}\,dx\,.$$

1. $I = \frac{1}{2}\Big(4 + \frac{1}{8}\Big)\pi$

2. $I = \frac{1}{4}\Big(2 - \frac{1}{8}\pi\Big)$

3. $I = 2 + \frac{1}{8}\pi$ **correct**

4. $I = 2 - \frac{1}{8}\pi$

5. $I = 4 - \frac{1}{8}\pi$

Explanation:
Set $x = 2\tan u$. Then

$$dx = 2\sec^2 u\, du,$$

while

$$x = 0 \implies u = 0,$$
$$x = 2 \implies u = \frac{\pi}{4}.$$

In this case

$$I = 2\int_0^{\pi/4} \frac{(4\tan^2 u + 5)\sec^2 u}{4\sec^2 u}\, du = \frac{1}{2}\int_0^{\pi/4} (4\tan^2 u + 5)\, du\,.$$

As with so many uses of substitution to evaluate an integral, this example requires further simplification before it can be computed. Since

$$\tan^2 u = \sec^2 u - 1, \quad \frac{d}{du}\tan u = \sec^2 u,$$

the integral can thus be rewritten as

$$I = \frac{1}{2}\int_0^{\pi/4} (4\sec^2 u + 1)\, du = \frac{1}{2}\Big[4\tan u + u\Big]_0^{\pi/4}.$$

Consequently,

$$\boxed{I = 2 + \frac{1}{8}\pi}.$$

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006 10.0 points

To which of the following does the integral

$$I = \int \frac{x^5}{\sqrt{1-x^2}}\, dx$$

reduce after an appropriate trig substitution?

1. $I = \int \sin^5(\theta)\, d\theta$ **correct**

2. $I = \int \tan(\theta)\sec^5(\theta)\, d\theta$

3. $I = \int \sin^5(\theta)\sec^6(\theta)\, d\theta$

4. $I = \int \sin^5(\theta)\sec^5(\theta)\, d\theta$

5. $I = \int \sec^5(\theta)\sin^6(\theta)\, d\theta$

Explanation:
Set $x = \sin(\theta)$. Then

$$dx = \cos(\theta)\, d\theta, \quad \sqrt{1-\sin^2(\theta)} = \cos(\theta)\,.$$

In this case

$$I = \int \frac{\sin^5(\theta)}{\cos(\theta)}\cos(\theta)\, d\theta\,.$$

Consequently,

$$\boxed{I = \int \sin^5(\theta)\, d\theta}.$$

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007 10.0 points

To which one of the following does the integral

$$I = \int \frac{x^2}{\sqrt{x^2+1}}\, dx$$

reduce after an appropriate trig substitution?

1. $I = \int \sec^3(\theta)\, d\theta$

2. $I = \int \tan^3(\theta)\, d\theta$

3. $I = \int \tan^2(\theta)\sec^3(\theta)\, d\theta$

4. $I = \int \sin^2(\theta)\sec^3(\theta)\, d\theta$ **correct**

5. $I = \int \sin^3(\theta)\, d\theta$

6. $I = \int \sin^3(\theta)\sec^2(\theta)\, d\theta$

Explanation:

Set $x = \tan(\theta)$. Then

$$dx = \sec^2(\theta)\, d\theta, \quad \sqrt{\tan^2(\theta)+1} = \sec(\theta)\,.$$

In this case,

$$I = \int \frac{\tan^2(\theta)}{\sec(\theta)}\sec^2(\theta)\, d\theta = \int \tan^2(\theta)\sec(\theta)\, d\theta\,.$$

Consequently,

$$\boxed{I = \int \sin^2(\theta)\sec^3(\theta)\, d\theta}\,.$$

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008 10.0 points

Evaluate the integral

$$I = \int_0^{1/4} \frac{3}{\sqrt{1-4x^2}}\, dx\,.$$

1. $I = \frac{3}{8}\pi$

2. $I = \frac{1}{2}\pi$

3. $I = \frac{1}{4}\pi$ **correct**

4. $I = \frac{1}{2}$

5. $I = \frac{3}{8}$

6. $I = \frac{1}{4}$

Explanation:

Set $2x = \sin(u)$. Then

$$2\, dx = \cos(u)\, du$$

and

$$1 - 4x^2 = 1 - \sin^2(u) = \cos^2(u)\,,$$

while

$$x = 0 \implies u = 0,$$

$$x = \frac{1}{4} \implies u = \frac{\pi}{6}.$$

In this case,

$$I = \frac{1}{2}\int_0^{\pi/6} \frac{3\cos(u)}{\cos(u)}\, du = \frac{3}{2}\int_0^{\pi/6} du\,.$$

Consequently,

$$\boxed{I = \frac{1}{4}\pi}\,.$$

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009 10.0 points

Evaluate the integral

$$I = \int_0^1 \frac{1}{\sqrt{4-3x^2}}\, dx\,.$$

1. $I = \frac{1}{3}$

2. $I = \frac{\pi}{3\sqrt{3}}$ **correct**

3. $I = 2$

4. $I = \frac{\frac{1}{2}\pi}{\sqrt{3}}$

5. $I = \frac{2\pi}{3\sqrt{3}}$

6. $I = \frac{1}{2}$

Explanation:

Set $\sqrt{3}x = 2\sin\theta$; then

$$\sqrt{3}\,dx \;=\; 2\cos\theta\,d\theta$$

and

$$4 - 3x^2 \;=\; 4(1-\sin^2\theta) \;=\; 4\cos^2\theta\,,$$

while

$$x = 0 \implies \theta = 0,$$

$$x = 1 \implies \theta = \frac{\pi}{3}.$$

In this case

$$I \;=\; \frac{2}{\sqrt{3}}\int_0^{\pi/3} \frac{\cos\theta}{2\cos\theta}\,d\theta \;=\; \frac{1}{\sqrt{3}}\int_0^{\pi/3} d\theta\,.$$

Consequently

$$\boxed{I \;=\; \frac{\pi}{3\sqrt{3}}}\,.$$

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010 10.0 points

Evaluate the integral

$$I \;=\; \int_0^1 \frac{3}{\sqrt{x^2+1}}\,dx\,.$$

1. $I \;=\; \sqrt{2}\,(\sqrt{2}-1)$

2. $I \;=\; 3\ln(1+\sqrt{2})$ **correct**

3. $I \;=\; 3\,(\sqrt{2}-1)$

4. $I \;=\; \sqrt{2}\,(1+\sqrt{2})$

5. $I \;=\; 3\ln(\sqrt{2}-1)$

6. $I \;=\; \sqrt{2}\,\ln(1+\sqrt{2})$

Explanation:

Set $x = \tan(u)$, then

$$dx \;=\; \sec^2(u)\,du\,, \quad x^2+1 \;=\; \sec^2(u)\,,$$

while

$$x = 0 \implies u = 0\,,$$

$$x = 1 \implies u = \frac{\pi}{4}.$$

In this case

$$I \;=\; \int_0^{\pi/4} \frac{3\sec^2(u)}{\sec(u)}\,du \;=\; \int_0^{\pi/4} 3\sec(u)\,du\,.$$

On the other hand,

$$\int \sec(u)\,du \;=\; \ln\left(|\sec(u)+\tan(u)|\right) + C\,.$$

Thus

$$I \;=\; 3\Big[\ln\left(|\sec(u)+\tan(u)|\right)\Big]_0^{\pi/4}.$$

Consequently,

$$\boxed{I \;=\; 3\ln(1+\sqrt{2})}\,.$$

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011 10.0 points

Evaluate the integral

$$I \;=\; \int_0^2 (6 - \sqrt{4-x^2}\,)\,dx\,.$$

1. $I \;=\; 6+\pi$

2. $I \;=\; 6+2\pi$

3. $I \;=\; 12-\pi$ **correct**

4. $I \;=\; 12-2\pi$

5. $I \;=\; 12+2\pi$

6. $I \;=\; 6-\pi$

Explanation:

Since

$$I = \int_0^2 6\,dx - \int_0^2 \sqrt{4-x^2}\,dx = I_1 + I_2,$$

we evaluate the two integrals separately. Now

$$I_1 \;=\; \int_0^2 6\,dx \;=\; 12\,.$$

On the other hand, to evaluate the second integral we set $x = 2\sin(u)$. For then

$$dx = 2\cos(u)\,du\,, \quad 4 - x^2 = 4\cos^2(u)\,,$$

while

$$x = 0 \implies u = 0\,,$$

$$x = 2 \implies u = \frac{\pi}{2}\,.$$

In this case

$$I_2 = 4\int_0^{\pi/2} \cos^2(u)\,du = 2\int_0^{\pi/2} (1 + \cos(2u))\,du\,.$$

Thus

$$I_2 = 2\Big[u + \frac{1}{2}\sin(2u)\Big]_0^{\pi/2} = \pi\,.$$

Consequently,

$$\boxed{I = I_1 - I_2 = 12 - \pi}\,.$$

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012 10.0 points

Evaluate the integral

$$I = \int_0^1 \frac{x^2}{1+x^2}\,dx\,.$$

1. $I = \frac{1}{4}(4 - \pi)$ **correct**

2. $I = \frac{1}{8}(4 - \pi)$

3. $I = \frac{1}{4}(4 + \pi)$

4. $I = \frac{1}{8}(\pi - 2)$

5. $I = \frac{1}{8}(\pi + 2)$

6. $I = \frac{1}{4}(\pi - 2)$

Explanation:

Let $x = \tan(\theta)$; then

$$dx = \sec^2(\theta)\,d\theta, \quad 1 + x^2 = \sec^2(\theta)\,,$$

while

$$x = 0 \implies \theta = 0,$$

$$x = 1 \implies \theta = \frac{\pi}{4}\,.$$

In this case,

$$I = 1\int_0^{\pi/4} \frac{\tan^2(\theta)}{\sec^2(\theta)}\sec^2(\theta)\,d\theta = 1\int_0^{\pi/4} \tan^2(\theta)\,d\theta\,.$$

But $\tan^2(\theta) = \sec^2(\theta) - 1$, so

$$I = 1\int_0^{\pi/4} \big(\sec^2(\theta) - 1\big)\,d\theta = 1\Big[\tan(\theta) - \theta\Big]_0^{\pi/4}\,.$$

Consequently

$$\boxed{I = \frac{1}{4}(4 - \pi)}\,.$$