This print-out should have 10 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Rewrite the expression

$$f(x) \;=\; \frac{2x}{x^2-3x+2}$$

using partial fractions.

1. $f(x) \;=\; \dfrac{2}{x-2}+\dfrac{1}{x-1}$
2. $f(x) \;=\; \dfrac{2}{x-2}+\dfrac{4}{x+1}$
3. $f(x) \;=\; \dfrac{4}{x-2}-\dfrac{2}{x-1}$ **correct**
4. $f(x) \;=\; \dfrac{2}{x-2}-\dfrac{1}{x-1}$
5. $f(x) \;=\; \dfrac{2}{x-2}-\dfrac{4}{x+1}$

Explanation:
Since

$$x^2-3x+2 \;=\; (x-2)(x-1),$$

we have to choose A, B so that

$$\frac{2x}{x^2-3x+2} \;=\; \frac{A}{x-2}+\frac{B}{x-1} \;=\; \frac{A(x-1)+B(x-2)}{(x-2)(x-1)}.$$

Equating numerators we thus see that

$$2x \;=\; A(x-1)+B(x-2) \;=\; x(A+B)-(A+2B).$$

Hence

$$A+B \;=\; 2, \quad A+2B \;=\; 0,$$

in which case

$$A \;=\; 4, \quad B \;=\; -2.$$

Consequently,

$$\boxed{f(x) \;=\; \frac{4}{x-2}-\frac{2}{x-1}}.$$

002 10.0 points

Rewrite the expression

$$f(x) \;=\; \frac{28}{x^2+x-12}$$

using partial fractions.

1. $f(x) \;=\; \dfrac{4}{x-3}-\dfrac{4}{x+4}$ **correct**
2. $f(x) \;=\; \dfrac{4}{x-3}+\dfrac{4}{x+4}$
3. $f(x) \;=\; \dfrac{5}{x-3}+\dfrac{3}{x+4}$
4. $f(x) \;=\; \dfrac{5}{x-3}-\dfrac{3}{x+4}$
5. None of these

Explanation:
Since

$$x^2+x-12 \;=\; (x-3)(x+4),$$

we have to choose A, B so that

$$\frac{28}{x^2+x-12} \;=\; \frac{A}{x-3}+\frac{B}{x+4} \;=\; \frac{A(x+4)+B(x-3)}{(x-3)(x+4)}.$$

Equating numerators we thus see that

$$28 \;=\; A(x+4)+B(x-3) \;=\; x(A+B)+(4A-3B).$$

Hence

$$A+B \;=\; 0, \quad 4A-3B \;=\; 28,$$

in which case

$$A \;=\; 4, \quad B \;=\; -4.$$

Consequently,

$$\boxed{f(x) \;=\; \frac{4}{x-3} - \frac{4}{x+4}}\,.$$

003 10.0 points

Rewrite the expression

$$f(x) \;=\; \frac{3x-2}{x^2(x-3)}$$

using partial fractions.

1. $f(x) \;=\; -\dfrac{7}{9x} + \dfrac{2}{3x^2} + \dfrac{7}{9(x-3)}$ **correct**

2. $f(x) \;=\; \dfrac{2}{3x} - \dfrac{7}{9x^2} - \dfrac{7}{9(x-3)}$

3. $f(x) \;=\; \dfrac{2}{x^2} - \dfrac{7}{x-3}$

4. $f(x) \;=\; -\dfrac{2}{x^2} + \dfrac{7}{x-3}$

5. $f(x) \;=\; \dfrac{7}{x} - \dfrac{2}{3x^2} + \dfrac{7}{9(x-3)}$

Explanation:

We have to find $A,\ B,$ and C so that

$$\frac{3x-2}{x^2(x-3)} \;=\; \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

$$=\; \frac{Ax(x-3) + B(x-3) + Cx^2}{x^2(x-3)}\,.$$

Thus

$$3x-2 \;=\; Ax(x-3) + B(x-3) + Cx^2\,.$$

Now

$$x \;=\; 0 \quad\Longrightarrow\quad B \;=\; \frac{2}{3},$$

while

$$x \;=\; 3 \quad\Longrightarrow\quad C \;=\; \frac{7}{9}\,.$$

But then on comparing coefficients of x^2 we see that

$$A + C \;=\; 0 \quad\Longrightarrow\quad A \;=\; -\frac{7}{9}.$$

Consequently,

$$\boxed{f(x) \;=\; -\frac{7}{9x} + \frac{2}{3x^2} + \frac{7}{9(x-3)}}\,.$$

004 10.0 points

Rewrite the expression

$$f(x) \;=\; \frac{9x}{(x-1)\,(x^2+x+1)}$$

using partial fractions.

1. $f(x) \;=\; \dfrac{3}{x-1} + \dfrac{3-3x}{x^2+x+1}$ **correct**

2. $f(x) \;=\; \dfrac{3}{x-1} + \dfrac{6x+9}{x^2+x+1}$

3. $f(x) \;=\; -\dfrac{6}{x-1} + \dfrac{6-3x}{x^2+x+1}$

4. $f(x) \;=\; \dfrac{6}{x-1} + \dfrac{6-3x}{x^2+x+1}$

5. $f(x) \;=\; -\dfrac{3}{x-1} - \dfrac{3+3x}{x^2+x+1}$

6. $f(x) \;=\; -\dfrac{3}{x-1} + \dfrac{6x+9}{x^2+x+1}$

Explanation:

We have to find $A,\ B$ and C so that

$$\frac{9x}{(x-1)\,(x^2+x+1)} \;=\; \frac{A}{x-1} + \frac{B\,x + C}{x^2+x+1}\,.$$

After bringing the right hand side to a common denominator and equating numerators, we thus see that

$$\begin{aligned} 9x \;&=\; A(x^2+x+1) + (Bx+C)(x-1) \\ &=\; A(x^2+x+1) \\ &\qquad + (Bx^2 + Cx - Bx - C) \\ &=\; (A+B)x^2 + (A+C-B)x \\ &\qquad + (A-C)\,. \end{aligned}$$

Now equate coefficients:

$$A + B \;=\; 0\,, \tag{1}$$
$$A + C - B \;=\; 9\,, \tag{2}$$
$$A - C \;=\; 0\,. \tag{3}$$

Adding all the three equations, we get:

$$3A = 9 \quad i.e., \quad A = 3.$$

Then, from (3) it follows that

$$C = A = 3,$$

and, finally, from (1),

$$B = -A = -3.$$

Consequently,

$$\boxed{f(x) = \frac{3}{x-1} + \frac{3-3x}{x^2+x+1}}.$$

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005 10.0 points

In the partial fractions decomposition of the expression

$$f(x) = \frac{x^3+2x-3}{x^2-x-2},$$

find the term having denominator $x-2$.

1. $-\dfrac{3}{x-2}$

2. $\dfrac{3}{x-2}$ **correct**

3. $-\dfrac{2}{x-2}$

4. $\dfrac{1}{x-2}$

5. $\dfrac{2}{x-2}$

6. $-\dfrac{1}{x-2}$

Explanation:

As $f(x) = P(x)/Q(x)$ with $\deg P \geq \deg Q$, we begin with long division:

$$\begin{array}{r|llll} & x & +1 & & \\ \hline x^2-x-2 & x^3 & +0x^2 & +2x & -3 \\ & x^3 & -x^2 & -2x & \\ \hline & & x^2 & +4x & -3 \\ & & x^2 & -x & -2 \\ \hline & & & 5x & -1 \end{array}$$

Thus

$$f(x) = x+1+\frac{5x-1}{x^2-x-2}.$$

On the other hand,

$$x^2-x-2 = (x-2)(x+1),$$

so we look for B and C satisfying

$$\frac{5x-1}{x^2-x-2} = \frac{B}{x-2} + \frac{C}{x+1}.$$

Multiplying through by $(x-2)(x+1)$ gives

$$5x-1 = B(x+1)+C(x-2).$$

so after substituting $x = 2$ and $x = -1$, we see that

$$B = 3, \qquad C = 2.$$

Consequently, $f(x)$ has partial fraction decomposition

$$\boxed{f(x) = x+1+\frac{3}{x-2}+\frac{2}{x+1}}.$$

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keywords: partial fractions

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006 10.0 points

Determine the indefinite integral

$$I = \int \frac{x+8}{(x+3)(x-2)}\,dx.$$

1. $I = \ln\left(\dfrac{(x-2)^2}{x+3}\right) + C$

2. $I = \ln\left(\dfrac{x+3}{(x-2)^2}\right) + C$

3. $I = \ln\left(\dfrac{(x-2)^2}{|x+3|}\right) + C$ **correct**

4. $I = \ln\left(\left|\dfrac{x-2}{x+3}\right|\right) + C$

5. $I = \ln\left(\frac{|x+3|}{(x-2)^2}\right) + C$

Explanation:

First we have to determine the partial fraction decomposition

$$\frac{x+8}{(x+3)\,(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}\,.$$

Multiply through by $(x+3)(x-2)$. Then

$$x+8 = A(x-2) + B(x+3)\,.$$

Setting $x = 2$ gives $10 = 5B$, *i.e*, $B = 2$, while setting $x = -3$ gives $5 = -5A$, *i.e.*, $A = -1$. Thus,

$$I = \int \left(-\frac{1}{x+3} + \frac{2}{x-2}\right) dx$$
$$= -\ln(|x+3|) + 2\ln(|x-2|) + C\,.$$

Consequently,

$$\boxed{I = \ln\left(\frac{(x-2)^2}{|x+3|}\right) + C}$$

with C an arbitrary constant.

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007 10.0 points

Evaluate the integral

$$I = \int_0^1 \frac{4}{(x+1)(x^2+1)}\,dx\,.$$

1. $I = 2\left(\frac{\pi}{2} - \ln(2)\right)$

2. $I = \ln(2) + \frac{\pi}{2}$ **correct**

3. $I = 2\left(\ln(8) - \frac{\pi}{2}\right)$

4. $I = 2\left(\ln(2) + \frac{\pi}{2}\right)$

5. $I = \frac{\pi}{2} - \ln(2)$

6. $I = \ln(8) - \frac{\pi}{2}$

Explanation:

By partial fractions,

$$\frac{4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}\,.$$

To determine A, B, and C multiply through by $(x+1)(x^2+1)$: for then

$$4 = A(x^2+1) + (x+1)(Bx+C)$$
$$= (A+B)x^2 + (B+C)x + (A+C)\,,$$

which after comparing coefficients gives

$$A = -B\,, \quad B = -C\,, \qquad A = 2\,.$$

Thus

$$I = 2\int_0^1 \left(\frac{1}{x+1} - \frac{x-1}{x^2+1}\right) dx$$
$$= 2\Big(\int_0^1 \frac{1}{x+1}\,dx - \int_0^1 \frac{x}{x^2+1}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx\Big)\,,$$

and so

$$I = 2\Big[\ln(x+1) - \frac{1}{2}\ln(x^2+1) + \tan^{-1}(x)\Big]_0^1$$
$$= \Big[\ln\frac{(x+1)^2}{x^2+1} + 2\tan^{-1}(x)\Big]_0^1\,.$$

Consequently,

$$\boxed{I = \ln(2) + \frac{\pi}{2}}\,.$$

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008 10.0 points

Evaluate the integral

$$I = \int_3^5 \frac{1}{(x-2)(6-x)}\,dx\,.$$

1. $I = \frac{1}{4}\ln(9)$ **correct**

2. $I = \ln(9)$

3. $I = \frac{1}{4}\ln\left(\frac{15}{7}\right)$

4. $I = \ln\left(\frac{15}{7}\right)$

5. $I = \frac{1}{3}\ln(9)$

6. $I = \frac{1}{3}\ln\left(\frac{15}{7}\right)$

Explanation:
To find A, B so that

$$\frac{1}{(x-2)(6-x)} = \frac{A}{x-2} + \frac{B}{6-x},$$

we first bring the right hand side to a common denominator. In this case,

$$\frac{1}{(x-2)(6-x)} = \frac{A(6-x)+B(x-2)}{(x-2)(6-x)},$$

and so

$$A(6-x)+B(x-2) = 1.$$

To find the values of A, and B, we can make particular choices of x:

$$x = 2 \implies A = \frac{1}{4},$$

and

$$x = 6 \implies B = \frac{1}{4}.$$

Thus

$$I = \int_3^5 \frac{1}{4}\left(\frac{1}{x-2} + \frac{1}{6-x}\right) dx.$$

Hence after integration,

$$I = \left[\frac{1}{4}\left(\ln(x-2) - \ln(6-x)\right)\right]_3^5$$

$$= \frac{1}{4}\left[\ln\left(\frac{x-2}{6-x}\right)\right]_3^5.$$

Consequently,

$$\boxed{I = \frac{1}{4}\ln(9)}.$$

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009 10.0 points

Evaluate the definite integral

$$I = \int_0^1 \frac{2x^2-3x+4}{x^2-x-2}\, dx\,.$$

1. $I = 2 - 4\ln 2$

2. $I = 3 + 5\ln 2$

3. $I = 2 - 5\ln 2$ **correct**

4. $I = 2 + 4\ln 2$

5. $I = 3 + 4\ln 2$

6. $I = 3 - 5\ln 2$

Explanation:
By division,

$$\frac{2x^2-3x+4}{x^2-x-2} = \frac{2(x^2-x-2)-x+8}{x^2-x-2} = 2 - \frac{x-8}{x^2-x-2}\,.$$

But by partial fractions,

$$\frac{x-8}{x^2-x-2} = \frac{3}{x+1} - \frac{2}{x-2}\,.$$

Thus

$$I = \int_0^1 \left\{2 - \frac{3}{x+1} + \frac{2}{x-2}\right\} dx\,.$$

Now

$$\int_0^1 \frac{3}{x+1}\, dx = \Big[3\ln|x+1|\Big]_0^1,$$

while

$$\int_0^1 \frac{2}{x-2}\,dx = \Big[2\ln|x-2|\Big]_0^1.$$

Consequently,

$$\boxed{I = \left[2x - \ln\left|\frac{(x+1)^3}{(x-2)^2}\right|\right]_0^1 = 2 - 5\ln 2}.$$

010 10.0 points

Find the unique function y satisfying the equations

$$\frac{dy}{dx} = \frac{6}{(x-2)(7-x)}, \quad y(3) = 0.$$

1. $y = \frac{6}{5}\left(\ln\left(\left|\frac{x-2}{7-x}\right|\right) + \ln(4)\right)$ **correct**
2. $y = \frac{6}{5}\left(\ln\left(\left|\frac{7-x}{x-2}\right|\right) - \ln(4)\right)$
3. $y = 6\left(\ln\left(\left|\frac{7-x}{x-2}\right|\right) - \ln(4)\right)$
4. $y = \frac{1}{5}\left(\ln\left(\left|\frac{x-2}{7-x}\right|\right) + \ln(4)\right)$
5. $y = 6\left(\ln\left(\left|\frac{x-2}{7-x}\right|\right) + \ln(4)\right)$

Explanation:

We first find A, B so that

$$\frac{6}{(x-2)(7-x)} = \frac{A}{x-2} + \frac{B}{7-x}$$

by bringing the right hand side to a common denominator. In this case,

$$\frac{6}{(x-2)(7-x)} = \frac{A(7-x)+B(x-2)}{(x-2)(7-x)},$$

and so

$$A(7-x) + B(x-2) = 6.$$

To find the values of A, B particular choices of x are made When $x = 2$, for instance, $A = \frac{6}{5}$, while when $x = 7$, $B = \frac{6}{5}$. Thus

$$\frac{dy}{dx} = \frac{6}{5}\Big(\frac{1}{x-2} + \frac{1}{7-x}\Big).$$

Hence after integration,

$$\begin{aligned} y &= \frac{6}{5}(\ln(|x-2|) - \ln(|7-x|)) + C \\ &= \frac{6}{5}\ln\left(\left|\frac{x-2}{7-x}\right|\right) + C \end{aligned}$$

with C an arbitrary constant. But

$$y(3) = 0 \implies C = -\frac{6}{5}\ln\Big(\frac{1}{4}\Big),$$

i.e.,

$$C = \frac{6}{5}\ln(4).$$

Consequently,

$$\boxed{y = \frac{6}{5}\left(\ln\left(\left|\frac{x-2}{7-x}\right|\right) + \ln(4)\right).}$$