Question #1

Rewrite the expression $f(x) = \frac{2x}{x^2-3x+2}$ using partial fractions.

$$f(x) = \frac{2x}{(x-1)(x-2)} = \frac{\overset{-2}{A}}{x-1} + \frac{\overset{4}{B}}{x-2}$$

$$A(x-2) + B(x-1)$$
$$Ax - 2A + Bx - B$$
$$2x = (A+B)x - (2A+B)$$
$$A+B = 2 \qquad \left[\begin{array}{cc|c} 1 & 1 & 2 \\ -2 & 1 & 0 \end{array}\right]$$
$$-2A - B = 0$$
$$+B \quad -2A = B$$

Question #2

$$\frac{\overset{4}{A}(x+4)}{(x-3)(x+4)} + \frac{\overset{-4}{B}(x-3)}{(x+4)(x-3)}$$

$$Ax + 4A + Bx - 3B$$
$$Ax + Bx + 4A - 3B$$
$$(A+B)x + 4A - 3B = 28$$
$$A + B = 0$$
$$4A - 3B = 28$$

Question #3

Rewrite the expression $f(x) = \frac{3x-2}{x^2(x-3)}$ using partial fractions.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} \qquad A = -7/9,\; B = 2/3,\; C = 7/9$$

$$Ax(x-3) + B(x-3) + C(x^2)$$
$$\underline{Ax^2} - \underline{3Ax} + \underline{Bx} - 3B + \underline{Cx^2}$$
$$(A+C)x^2 + (-3A+B)x - 3B = 3x - 2$$
$$A + C = 0$$
$$-3A + B = 3$$
$$-3B = -2$$

Question #9: Evaluate the definite integral

$$I = \int_0^1 \frac{2x^2-3x+4}{x^2-x-2}\,dx \qquad x^2-x-2 \overline{\big)\,2x^2-3x+4}^{\;2} \quad \frac{-\,2x^2+2x+4}{-x+8}$$

$$2 + \int \frac{-x+8}{(x+1)(x-2)}\,dx \qquad \begin{array}{r} x+1 \\ \times\; x^2 \quad x \\ -2 \quad -2x \quad -2 \\ \hline x^2-x-2 \end{array}$$

$$(x+1)(x-2)(I) = \left(\frac{A}{x+1} + \frac{B}{x-2}\right)(x+1)(x-2)$$

$$A(x-2) + B(x+1)$$
$$Ax - 2A + Bx + B$$
$$(A+B)x + (-2A+B)$$

$$A + B = -1 \qquad \left[\begin{array}{cc|c} 1 & 1 & -1 \\ -2 & 1 & 8 \end{array}\right] R_2+2R_1$$
$$-2A + B = 8 \qquad \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 3 & 6 \end{array}\right] \tfrac{1}{3}R_2$$
$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 2 \end{array}\right] R_1 - R_2$$
$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right] \begin{array}{l} A \\ B \end{array}$$

$$-3\int \frac{1}{x+1} + 2\int \frac{1}{x-2}$$
$$2\ln|x-2| - 3\ln|x+1| \Big]_0^1$$
$$\left(2\ln|-1| - 3\ln|2|\right) - \left(2\ln|-2| - 3\ln|1|\right)$$
$$-3\ln|2| - 2\ln|-2|$$
$$-3\ln|2| - 2\ln|2|$$
$$(\ln|2|)\; -3 - 2 + 2$$
$$\boxed{2 - 5(\ln|2|) = I}$$

$2 - 0$

Question #10: Find the unique function y satisfying the equation

$$\frac{dy}{dx} = \frac{6}{(x-2)(7-x)}, \quad y(3) = 0$$

$$(x-2)(7-x)\frac{6}{(x-2)(7-x)} = \left(\frac{A}{x-2} + \frac{B}{7-x}\right)(x-2)(7-x)$$

$$\frac{6}{5}\int \frac{1}{x-2} + \frac{1}{7-x} \qquad 6 = A(7-x) + B(x-2)$$
$$\frac{6}{5}\int \frac{1}{x-2} - \frac{1}{x-7} \qquad = 7A - Ax + Bx - 2B$$
$$\frac{6}{5}\left(\ln|x-2| - \ln|x-7|\right) \qquad (B-A)x + (7A-2B)$$
$$B - A = 0$$
$$7A - 2B = 6$$
$$y = \frac{6}{5}\ln\left(\left|\frac{x-2}{x-7}\right|\right) + C$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 7 & -2 & 6 \end{array}\right] R_2 + 7R_1 \qquad -2 + 7 = 5$$
$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 5 & 6 \end{array}\right] \tfrac{1}{5}R_2$$
$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 1 & 6/5 \end{array}\right] -R_1$$
$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 6/5 \end{array}\right] R_1 + R_2$$
$$\left[\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 6/5 \end{array}\right] \begin{array}{l} A \\ B \end{array}$$

Question #4

Rewrite the expression $f(x) = \frac{9x}{(x-1)(x^2+x+1)}$ using partial fractions.

$$(x-1)(x^2+x+1)\left(\frac{9x}{(x-1)(x^2+x+1)}\right) = \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\right)(x-1)(x^2+x+1)$$

$$A(x^2+x+1) + Bx + C(x-1) \qquad \begin{array}{r} Bx + C \\ \times\; Bx^2 \quad Cx \\ 1 \quad -Bx \quad -C \end{array}$$
$$9x = \left(Ax^2 + Ax + A\right) + \left(Bx^2 - Bx + Cx - C\right)$$
$$\underline{Ax^2} + \underline{Ax} + \underline{A} + \underline{Bx^2} - \underline{Bx} + Cx - \underline{C}$$
$$(A+B)x^2 + (A-B+C)x + (A-C)$$

$A = 3$

$B = -3$

$C = 3$

$$A + B = 0 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 9 \\ 1 & 0 & -1 & 0 \end{array}\right] \begin{array}{l} \\ -R_2 \\ -R_3 \end{array}$$
$$A - B + C = 9$$
$$A - C = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & -9 \\ -1 & 0 & 1 & 0 \end{array}\right] R_2 - R_3 \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -9 \\ 1 & 0 & 1 & 0 \end{array}\right] R_3 - R_1 \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -9 \\ 0 & -1 & 1 & 0 \end{array}\right] R_3 + R_2 \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -9 \\ 0 & 0 & -1 & -9 \end{array}\right] R_2 - 2R_3$$

$-9 + 18 = 9$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -9 \end{array}\right] R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -9 \end{array}\right]$$

$$\frac{3}{x-1} + \frac{-3x+3}{x^2+x+1} \quad \text{or} \quad \frac{3}{x-1} + \frac{3-3x}{x^2+x+1}$$
$$\text{or} \quad \frac{3}{x-1} - \frac{3x-3}{x^2+x+1}$$

Question #5:

In the partial fractions decomposition of this expression

$$f(x) = \frac{x^3+2x-3}{x^2-x-2} \qquad \begin{array}{r} x-2 \\ \times\; x^2 \quad -2x \\ +1 \quad x \quad -2 \end{array} \qquad \begin{array}{rr} 2 & -1 \\ -2 & 1 \end{array}$$

Find the term having denominator $x-2$.

$$(x+1)(x-2)\left(\frac{x^3+2x-3}{(x+1)(x-2)}\right) = \left(\frac{A}{x+1} + \frac{\overset{3}{B}}{x-2}\right)(x+1)(x-2) \quad x^2 - x - 2$$

$$A(x-2) + B(x+1) = x^3 + 2x - 3$$

$$\begin{array}{r} x+1 \\ x^2-x-2 \overline{\big)\,x^3+2x-3} \\ -\;x^3+x^2+2x \\ x^2+4x-3 \\ -\;x^2-x-2 \\ 5x-1 \end{array}$$

$$x+1 + \frac{5x-1}{x^2-x-2}$$

$$Ax - 2A + Bx + B$$
$$(A+B)x + (-2A+B) = 5x - 1$$
$$A + B : 5 \quad \Big| \quad A + B = 5$$
$$2A + B = 1 \quad \Big| \quad -2A + B = -1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & -1 & 1 \end{array}\right] R_2 - 2R_1 \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -3 & -9 \end{array}\right] -\tfrac{1}{3}R_2$$
$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \end{array}\right] R_1 - R_2$$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \begin{array}{l} A \\ B \end{array}$$

Question #6:

Determine the indefinite integral

$$I = \int \frac{x+8}{(x+3)(x-2)}\,dx.$$

$$(x+3)(x-2)(I) = \frac{A}{x+3} + \frac{B}{x-2}$$

$$x + 8 = A(x-2) + B(x+3)$$
$$Ax - 2A + Bx + 3B \qquad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & 3 & 8 \end{array}\right] R_2 + 2R_1$$
$$(A+B)x + (-2A+3B)$$
$$A + B = 1 \qquad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 5 & 10 \end{array}\right] \tfrac{1}{5}R_2$$
$$-2A + 3B = 8$$
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right] R_1 - R_2$$

$$I = \int \frac{-1}{x+3} + \frac{2}{x-2} \qquad \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right] \begin{array}{l} A \\ B \end{array}$$
$$I = -\int \frac{1}{x+3} + 2\int \frac{1}{x-2}$$

$$-\left(\ln|x+3|\right) + 2\ln|x-2|$$
$$-\ln|x+3| + 2\ln|x-2| + C$$
$$\ln|(x-2)^2| - \ln|x+3| + C$$
$$\ln\left|\frac{(x-2)^2}{x+3}\right| + C \equiv \ln\left(\frac{(x-2)^2}{|x+3|}\right) + C$$

$$w \cdot \log x = \log|(x)^w|$$
$$c \cdot \ln A = \ln|A^c|$$

$$\log A - \log B = \log\left|\frac{A}{B}\right|$$
$$\ln A - \ln B = \ln\left|\frac{A}{B}\right|$$

Question #7: Evaluate the integral

$$I = \int_0^1 \frac{4}{(x+1)(x^2+1)}\,dx.$$

$$((x+1)(x^2+1))I = \left(\frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right)(x+1)(x^2+1)$$

$$\begin{array}{r} Bx + C \\ Ax^2 + A + Bx^2 + Cx + Bx + C \qquad \times\; Bx^2 + Cx \\ +1 \quad Bx + C \end{array}$$
$$(A+B)x^2 + (C+B)x + (A+C)$$

$$A + B = 0 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 4 \end{array}\right] R_3 - R_1$$
$$C + B = 0$$
$$A + C = 4$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 4 \end{array}\right] R_3 + R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 4 \end{array}\right] \tfrac{1}{2}R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] R_2 - R_3$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} A \\ B \\ C \end{array}$$

$$\int_0^1 \frac{2}{x+1} + \int_0^1 \frac{-2x+2}{x^2+1}\,dx$$

$$2\ln|x+1| \qquad \int_0^1 \frac{-2x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

Let $u = x^2$

$du = 2x\,dx$

$$-\int_0^1 \frac{1}{u+1}$$
$$-\ln|u+1|$$

$$2\ln|x+1| - \ln|x^2+1| + 2\tan^{-1}(x)\Big]_0^1$$
$$\left(2\ln|2| - \ln|2| + 2\tan^{-1}(1)\right) - \left(\cancel{2\ln|1|} - \cancel{2\ln|1|} + \cancel{2\tan^{-1}(0)}\right)$$
$$2\ln|2| - \ln|2| + \frac{\pi}{2}$$
$$\ln|2^2| - \ln|2| + \frac{\pi}{2}$$
$$\left(\ln|4| - \ln|2|\right) + \frac{\pi}{2}$$
$$\boxed{\ln|2| + \frac{\pi}{2}}$$

Question #8: Evaluate the integral

$$I = \int_3^5 \frac{1}{(x-2)(6-x)}\,dx.$$

$$(x-2)(6-x)(I) = \left(\frac{A}{x-2} + \frac{B}{6-x}\right)(x-2)(6-x)$$

$$A(6-x) + B(x-2)$$
$$6A - Ax + Bx - 2B$$
$$(B-A)x + (6A-2B) \qquad 6 - 2 = 4$$

$$B - A = 0 \qquad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 6 & -2 & 1 \end{array}\right] R_2 + 6R_1$$
$$6A - 2B = 1$$
$$I = \int_3^5 \frac{(1/4)}{x-2} + \int_3^5 \frac{(1/4)}{6-x} \qquad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 4 & 1 \end{array}\right] R_2 - 3R_1$$

$3 - 2$

$$\frac{1}{4}\ln|x-2| + \frac{1}{4}\ln|6-x|$$
$$\frac{1}{4}\left(\ln|(x-2)(6-x)|\right)\Big]_3^5 \qquad 3(1)$$

3

$5 - 2 \qquad 6 - 5$

$$\left(\frac{1}{4}\ln|3|\right) \quad \cancel{\ln|1|} \to 0 \qquad \frac{1}{4}\ln|1| + \frac{1}{4}\ln|6-3|$$
$$\frac{1}{4}\ln|6-3|$$
$$\frac{1}{4}\ln|3| - \frac{1}{4}\ln|3|$$
$$\frac{1}{4}\ln\left|\frac{x-6}{x-2}\right|\Big]_3^5$$
$$\frac{1}{4}\left(\ln\left|\frac{3}{3}\right|\right)$$
$$\frac{1}{4}\ln\left|\frac{-1}{3}\right| - \frac{1}{4}\ln\left|\frac{-3}{1}\right|$$

$$\frac{1}{3} \cdot \frac{1}{3} \qquad \ln\left|\frac{1}{6}\right|$$