This print-out should have 5 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Which of the following integrals are improper?

$$I_1 = \int_0^1 \frac{1}{\sqrt{x}}\,dx\,,$$

$$I_2 = \int_0^2 \frac{x+2}{x+1}\,dx\,,$$

$$I_3 = \int_1^\infty \frac{1}{1+x^2}\,dx\,.$$

1. I_3 only

2. none of them

3. I_2 and I_3 only

4. I_2 only

5. I_1 and I_3 only **correct**

6. I_1 and I_2 only

7. I_1 only

8. all of them

Explanation:

An integral

$$I = \int_a^b f(x)\,dx$$

is improper when one or more of the following conditions are satisfied:

(i) the interval of integration is infinite, *i.e.*, when $a = -\infty$ or $b = \infty$, or both;

(ii) f has a vertical asymptote at one or more of $x = a$, $x = b$ or $x = c$ for some $a < c < b$.

Consequently,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

is improper;

$$\int_0^2 \frac{x+2}{x+1}\,dx$$

is not improper;

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

is improper.

002 10.0 points

Determine if the integral

$$I = \int_0^\infty e^{-9x}\,dx$$

is convergent, and if it is, find its value.

1. $I = -\dfrac{1}{9}$

2. $= 9$

3. $I = \dfrac{1}{9}$ **correct**

4. $I = -9$

5. integral is divergent

Explanation:

The integral is improper because of the infinite interval of integration. It will be convergent if

$$\lim_{t\to\infty} \int_0^t e^{-9x}\,dx$$

exists. Now

$$\int_0^t e^{-9x}\,dx \;=\; \left[-\frac{1}{9}e^{-9x}\right]_0^t$$
$$= -\frac{1}{9}e^{-9t} + \frac{1}{9}\,.$$

On the other hand,

$$\lim_{t\to\infty} e^{-9t} \;=\; 0\,.$$

Consequently, I exists and

$$\boxed{I \;=\; \frac{1}{9}}\,.$$

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003 10.0 points

Determine if the improper integral

$$I \;=\; \int_4^\infty e^{-x/2}\,dx$$

converges, and if it does, compute its value.

1. $I = e^{-2}$

2. $I = 2e^2$

3. I does not converge

4. $I = 2e^{-2}$ **correct**

5. $I = -2e^{-2}$

Explanation:

The integral

$$I = \int_4^\infty e^{-x/2}\,dx$$

is improper because of the infinite range of integration. To overcome this we restrict to a finite interval of integration and consider the limit

$$\lim_{n\to\infty} I_n, \quad I_n = \int_4^n e^{-x/2}\,dx.$$

If this limit exists, then we set

$$I = \lim_{n\to\infty}\int_4^n e^{-x/2}\,dx.$$

But

$$\int_4^n e^{-x/2}\,dx = \left[-2\,e^{-x/2}\right]_4^n$$
$$= \left\{2\,e^{-2} - 2\,e^{-n/2}\right\}.$$

Since $e^{-n/2} \to 0$ as $n \to \infty$, the $\lim\limits_{n\to\infty} I_n$ exists and

$$\boxed{I \;=\; \int_4^\infty e^{-x/2}\,dx \;=\; 2\,e^{-2}}\,.$$

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004 10.0 points

Determine if the improper integral

$$I \;=\; \int_4^\infty 2xe^{-4x^2}\,dx$$

converges, and if it does, find its value.

1. $I \;=\; \frac{1}{4}e^{-64}$ **correct**

2. $I \;=\; \frac{1}{4}e^{64}$

3. I does not converge

4. $I \;=\; 2e^{-64}$

5. $I \;=\; \frac{1}{2}e^{64}$

6. $I \;=\; \frac{1}{2}e^{-64}$

Explanation:

The integral

$$I \;=\; \int_4^\infty 2xe^{-4x^2}\,dx$$

is improper because of the infinite range of integration. To overcome this we restrict to a finite interval of integration and consider the limit

$$I = \lim_{t\to\infty} I_t, \quad I_t = \int_4^t 2xe^{-4x^2}\,dx.$$

To evaluate I_t we use the substitution $u = x^2$. For then

$$I_t = \int_{16}^{t^2} e^{-4u}\,du = -\frac{1}{4}\Big[e^{-4u}\Big]_{16}^{t^2} = \frac{1}{4}\Big(e^{-64} - e^{-4t^2}\Big).$$

But,

$$\lim_{x\to\infty} e^{-ax^2} = 0,$$

for any $a > 0$, so

$$\lim_{t\to\infty} e^{-4t^2} = 0.$$

Consequently, I converges and

$$\boxed{I = \frac{1}{4}e^{-64}}.$$

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005 10.0 points

Determine if the improper integral

$$I = \int_1^\infty \frac{6x}{(1+x^2)^2}\,dx$$

converges, and if it does, compute its value.

1. $I = 6$

2. $I = \frac{3}{2}$ **correct**

3. $I = 3$

4. $I = 2$

5. integral doesn't converge

Explanation:

The integral

$$I = \int_1^\infty \frac{6x}{(1+x^2)^2}\,dx$$

is improper because of the infinite interval of integration. To overcome this, we truncate and consider the limit

$$\lim_{t\to\infty} I_t, \quad I_t = \int_1^t \frac{6x}{(1+x^2)^2}\,dx\,.$$

To evaluate I_t, set $u = 1 + x^2$. Then

$$du = 2x\,dx\,,$$

in which case

$$\int \frac{6x}{(1+x^2)^2}\,dx = 3\int \frac{1}{u^2}\,du.$$

Thus

$$I_t = \int_1^t \frac{6x}{(1+x^2)^2}\,dx = 3\Big[-\frac{1}{1+x^2}\Big]_1^t = 3\Big\{\frac{1}{2} - \frac{1}{1+t^2}\Big\}.$$

Consequently, since

$$\lim_{t\to\infty} \frac{1}{1+t^2} = 0,$$

we see that I converges and that

$$\boxed{I = \lim_{t\to\infty} \int_1^t \frac{6x}{(1+x^2)^2}\,dx = \frac{3}{2}}.$$