Part I of Sec. 7.8

Question #1: Which of the following integrals are improper?

$\lim_{x\to 0^+} \int^1 \frac{1}{\sqrt{x}} dx$

1. $I_1 = \int_0^1 \frac{1}{\sqrt{x}} dx$
2. $I_2 = \int_0^2 \frac{x+2}{x+1}$
3. $I_3 = \int_1^\infty \frac{1}{1+x^2} dx$

Let $u = x$
$du = dx$

$\frac{1}{u^{1/2}} \to u^{-1/2}$

$\frac{1}{-\frac{1}{2}+1} u^{-1/2+1}$

$\int_0^1 2u^{1/2}$

$\left[2(1)^{1/2} - \cancel{2(t)^{1/2}}^{\,0}\right]_0^1$

$2 - 0 = 2$

1 and 3 are improper integrals since for I_1 if you plugin 0 zero into the integral

Question #2: Determine if the integral $I = \int_0^\infty e^{-9x} dx$ is convergent, and if it does, compute its value.

$\lim_{b\to\infty} \int_0^b e^{-9x} dx$

$\lim_{b\to\infty} -\frac{1}{9} e^{-9x} \Big]_0^b$

$-\frac{1}{9} e^{-9b} + \frac{1}{9} e^{-9(0)}$

$0 + \frac{1}{9}$

$\boxed{\frac{1}{9}}$

Question #3: Determine if the improper integral
$I = \int_4^\infty e^{-x/2} dx$
converges, and if it does, compute its value.

$I = \int_4^\infty e^{-\frac{x}{2}} dx$

Question #5: Determine if the improper integral
$I = \int_1^\infty \frac{6x}{(1+x^2)^2} dx$
converges, and if it does, compute its value.

$I = \int_1^\infty \frac{6x}{(1+x^2)^2} dx \to$ Let $u = 1 + x^2$
$du = 2x\, dx$
$\frac{1}{2} du = x\, dx$

$\int_1^\infty \frac{6 \cdot \frac{1}{2} du}{(u)^2} \to \frac{3du}{u^2}$

$3 \int_1^\infty \frac{1}{u^2} du$

$\frac{1}{-2+1} u^{-2+1}$

$-u^{-1}$

$3\left(-\frac{1}{u}\right)$

$\frac{-3}{u}$

$\frac{-3}{1+x^2}\Big]_1^\infty$

$\boxed{\frac{3}{2}}$

Question #4: $\int_4^\infty 2x e^{-4x^2} dx$

Let $u = x^2$
$du = 2x\, dx$

$\int e^{-4u} du$

$-\frac{1}{4} e^{-4u}$

$-\frac{1}{4} e^{-4x^2}\Big]_4^\infty = \cancel{-\frac{1}{4} e^{-4(\infty)^2}} \to 0 + \frac{1}{4} e^{-4(4)^2}$

Ignore

$\boxed{\frac{1}{4} e^{-64}}$ ✓

Part 2 of Sec. 7.8

Question #1: Determine if the improper integral

$I = \int_0^5 \frac{2\ln(x)}{x} dx$

$\lim_{c\to 0^+} \int_c^5 \frac{2\ln(x)}{x} dx$ $\qquad uv - \int v\, du$

$\lim_{c\to 0^+} 2\int_c^5 \frac{\ln(x)}{x}$

Let $u = \ln x$
$du = \frac{1}{x} dx$

$\int u\, du$

$\frac{1}{2} u^2$

$2\left(\frac{1}{2}(\ln x)^2\right)$

$\lim_{c\to 0^+} (\ln x)^2 \Big]_0^5$

$(\ln(5))^2 - \ln(c)^2$
$- (-\infty)$

$(\ln(5)) + \infty$

$(\ln(5))^2 - \left(\lim_{c\to 0^+} \ln(c)^2\right) = I$

$\boxed{\infty}$ (divergent)

Question #2: Determine if the improper integral
$I = \int_0^1 \frac{4\sin^{-1} x}{\sqrt{1-x^2}} dx$
is convergent or divergent, and if convergent find its value.

Since $\sqrt{a^2 - x^2} \to x = a \sin\theta$
This mean $x = \sin\theta$ for this integrand.

Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$\lim_{a\to 0^+} \int_a^1 \frac{4\sin^{-1}(\sin(\theta))}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$

And $\sin^2\theta + \cos^2\theta = 1$

$4\int \underbrace{\frac{\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta\, d\theta}$

$\frac{\theta}{\cancel{\cos\theta}} \cancel{\cos\theta}\, d\theta$

Since $x = \sin\theta$
$\arcsin(x) = \theta$

$4\int \theta\, d\theta \to \frac{1}{2}\theta^2$

$4\left(\frac{1}{2}\theta^2\right)$

$2\theta^2 \Rightarrow 2(\arcsin(x))^2\Big]_a^1$

$\lim_{a\to 0^+} 2(\arcsin(x))^2\Big]_a^1$

$2(\arcsin(1))^2 - \cancel{(2\arcsin(a))^2} \to 0$

$2\left(\frac{\pi}{2}\right)^2$

$2\left(\frac{\pi^2}{4}\right) \to \boxed{\frac{1}{2}\pi^2}$ Convergent!

Question #3: Determine if the integral

$I = \int_0^2 \frac{2}{(x-1)^{2/3}} dx$

is convergent or divergent; and if convergent, find its value.

$I = \int_0^2 \frac{2}{(x-1)^{2/3}} dx \to$ Discontinuity @ $x=1$ per the linear func.

$\to \lim_{b\to 1^-} 2\int_0^1 \frac{2}{(x-1)^{2/3}} + \lim_{a\to 1^+} \int_1^2 \frac{2}{(x-1)^{2/3}}$

$\lim_{b\to 1^-} 2\int_0^1 \frac{2}{(x-1)^{2/3}} dx + \lim_{a\to 1^+} \int_1^2 \frac{2}{(x-1)^{2/3}} dx$

Let $u = x - 1$
$du = dx$

$2\int \frac{1}{u^{2/3}} du$

$2\int u^{-2/3}$

$2(3u^{1/3})$

$6u^{1/3}$

App. $\to$

$\lim_{a\to 1^+} \left(6(x-1)^{1/3}\Big]_a^2\right)$

$\left(6(2-1)^{1/3}\right) - \cancel{\left(6(1-1)^{1/2}\right)} \to 0$

$\boxed{6}$

$\lim_{b\to 1^-} 6(x-1)^{1/3}\Big]_0^b$

$\lim_{b\to 1^-} \cancel{\left(6(b-1)^{1/3}\right)} \to 0 - \left(6(0-1)^{1/3}\right)\Big]_0^b$

$0 + 6 = \boxed{6}$

$$\boxed{\begin{matrix} 6 + 6 = \\ I_1 + I_2 = 12 \end{matrix}}$$

Convergent!