This print-out should have 3 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

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001 10.0 points

Determine if the improper integral

$$I = \int_0^5 \frac{2\ln(x)}{x}\,dx$$

converges, and if it does, compute its value.

1. $I = 2(\ln(5))^2$

2. $I = (\ln(5))^2$

3. I does not converge **correct**

4. $I = \ln(5)$

5. $I = 2\ln(5)$

Explanation:

The integral is improper because

$$\lim_{x \to 0^+} \frac{\ln(x)}{x} = -\infty,$$

so we have to consider

$$\lim_{t \to 0^+} I_t, \qquad I_t = \int_t^5 \frac{2\ln(x)}{x}\,dx.$$

To evaluate I_t, set $u = \ln(x)$. Then

$$du = \frac{dx}{x},$$

while

$$x = t \implies u = \ln(t),$$

$$x = 5 \implies u = \ln(5).$$

In this case

$$I_t = \int_{\ln(t)}^{\ln(5)} 2\,u\,du = \Big[u^2\Big]_{\ln(t)}^{\ln(5)}$$

$$= \Big((\ln(5))^2 - (\ln(t))^2\Big).$$

On the other hand,

$$\lim_{t \to 0^+} (\ln(t))^2 = \infty.$$

Consequently,

$$\boxed{I \text{ does not converge}}.$$

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002 10.0 points

Determine if the improper integral

$$I = \int_0^1 \frac{4\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$$

is convergent or divergent, and if convergent, find its value.

1. $I = \frac{1}{4}\pi^2$

2. I is divergent

3. $I = \frac{1}{2}\pi^2$ **correct**

4. $I = \frac{1}{4}$

5. $I = \frac{1}{2}$

Explanation:

The integral is improper because

$$\lim_{x \to 1-} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} = \infty.$$

Thus we have to check if

$$\lim_{t \to 1-} \int_0^t \frac{4\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$$

is convergent or divergent. To determine

$$\int \frac{4\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$$

set $u = \sin^{-1}(x)$. Then

$$du = \frac{1}{\sqrt{1-x^2}}\,dx,$$

and so

$$\int \frac{4\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx = 4\int u\,du = 2u^2 + C\,,$$

from which it follows that

$$\int_0^t \frac{4\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$$

$$= 2\Big[(\sin^{-1}(x))^2\Big]_0^t = 2(\sin^{-1}(t))^2\,.$$

Now

$$\lim_{t\to 1-} (\sin^{-1}(t))^2 = \frac{\pi^2}{4}\,.$$

Consequently, I is convergent and

$$\boxed{I = \frac{1}{2}\pi^2}\,.$$

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003 10.0 points

Determine if the integral

$$I = \int_0^2 \frac{2}{(x-1)^{2/3}}\,dx$$

is convergent or divergent; and if convergent, find its value.

1. $I = 0$

2. $I = 4$

3. $I = 12$ **correct**

4. $I = 6$

5. I is divergent

Explanation:

The integral is improper because the function

$$f(x) = \frac{2}{(x-1)^{2/3}}$$

has a vertical asymptote at $x = 1$ in the interval of integration. Thus we have to consider each of the improper integrals

$$I_1 = \int_0^1 \frac{2}{(x-1)^{2/3}}\,dx,$$

$$I_2 = \int_1^2 \frac{2}{(x-1)^{2/3}}\,dx$$

separately.

Now for $t < 1$,

$$\int_0^t \frac{2}{(x-1)^{2/3}}\,dx = \Big[6(x-1)^{1/3}\Big]_0^t$$

$$= 6(t-1)^{1/3} + 6\,.$$

But

$$\lim_{t\to 1-} (t-1)^{1/3} = 0,$$

so I_1 is convergent and

$$I_1 = \lim_{t\to 1-} 6\Big((t-1)^{1/3} + 1\Big) = 6\,.$$

On the other hand, for $t > 1$,

$$\int_t^2 \frac{2}{(x-1)^{2/3}}\,dx = \Big[6(x-1)^{1/3}\Big]_t^2$$

$$= 6 - 6(t-1)^{1/3}\,.$$

But

$$\lim_{t\to 1+} (t-1)^{1/3} = 0,$$

so I_2 is convergent and

$$I_2 = \lim_{t\to 1+} 6\Big(1 - (t-1)^{1/3}\Big) = 6\,.$$

Consequently, I is convergent and

$$\boxed{I = I_1 + I_2 = 12}\,.$$