Question #1 (Part I of 3):

For the differential equation

$$\frac{dy}{dx} + 2y = 8e^{3x}$$

(i) first find its general solution.

General equation equation form for a 1^{st} order linear differential equation:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

To solve multiply by integrating factor $I(x) = e^{\int P(x)\,dx}$ on both sides.

$$\frac{dy}{dx} + 2y = 8e^{3x} \quad \text{where } P(x) = 2$$

Thus, $\int 2\,dx \rightarrow 2x$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 8e^{3x}e^{2x}$$

$$\underbrace{e^{2x}}_{f}\underbrace{\frac{dy}{dx}}_{g'} + \underbrace{2e^{2x}}_{f'}\underbrace{y}_{g} = 8e^{5x}$$

$$\int \frac{d}{dx}(e^{2x}\cdot y) = \int 8e^{5x}\,dx$$

$$e^{2x}y + \cancel{C} = \frac{8}{5}e^{5x} + C$$
$$-\cancel{C} \qquad -C$$

$$y = \frac{\frac{8}{5}e^{5x} + C}{e^{2x}}$$

$$y = \frac{\frac{8}{5}e^{5x}}{e^{2x}} + \frac{C}{e^{2x}}$$

$$\boxed{y = \frac{8}{5}e^{3x} + Ce^{-2x}}$$

Question #2 (Part II of 3): Find the particular solution y_0 such that $y_0(0) = 8$.

Let: $x = 0$, $y = 8$

$$8 = \frac{8}{5}\cancel{e^{3(0)}}^{1} + C\cancel{e^{-2(0)}}^{1}$$

$$8 = \frac{8}{5} + C$$

$$\frac{40}{5} = \cancel{\frac{8}{5}} + C \qquad \frac{32}{5}$$

$$-\cancel{\frac{8}{5}}$$

$$-\frac{8}{5} + \frac{40}{5} = C$$

$$\boxed{y_0 = \frac{8}{5}e^{3x} + \frac{32}{5}e^{-2x}}$$

Question #3 (Part III of 3): For the particular solution y_0 in (ii), determine the value $y_0(1)$

$$y_0(1) = \frac{8}{5}e^{3x} + \frac{32}{5}e^{-2x}$$

$$y_0(1) = \frac{8}{5}e^{3(1)} + \frac{32}{5}e^{-2(1)} = \frac{8}{5}e^{3} + \frac{32}{5}e^{-2}$$

$$\boxed{y_0(1) = \frac{8}{5}e^{3} + \frac{32}{5}e^{-2}}$$

Question #4: If y_0 is the solution of the equations

$xy' + 2y = 4x$, $y(1) = 6$,

Determine the value of $y_0(2)$.

Let $x = 1$, $y = 6$

$$x\frac{dy}{dx} + 2y = 4x \qquad P(x) = \frac{2}{x}$$

$$\div x \qquad e^{\int P(x)\,dx}$$

$$(x^2)\left(\frac{dy}{dx} + \frac{2}{x}y\right) = 4(x^2) \qquad x^2 = e^{\ln x^2}$$

$$\underbrace{x^2}_{f}\underbrace{\frac{dy}{dx}}_{g'} + \underbrace{2x}_{f'}\underbrace{y}_{g} = 4x^2$$

$$\int \frac{d}{dx}(x^2 y) = \int 4x^2 \rightarrow 4\int x^2 \qquad \frac{1}{3}x^3$$

$$x^2 y + C = \frac{4}{3}x^3 + C$$
$$-C \qquad -C$$

$$\frac{x^2 y}{x^2} = \frac{\frac{4}{3}x^3 + C}{x^2}$$

$$y = \frac{4}{3}x + \frac{C}{x^2}$$

$$6 = \frac{4}{3}(1) + \frac{C}{1^2}$$

$$6 = \frac{4}{3} + C$$

$$-\frac{4}{3} + \frac{18}{3} = \cancel{\frac{4}{3}} + C$$

$$-\cancel{\frac{4}{3}}$$

$$C = \frac{14}{3}$$

$$3\overline{)56}$$

$$y_0 = \frac{4}{3}x + \frac{\frac{14}{3}}{x^2}$$

$$y_0(2) = \frac{4}{3}(2) + \frac{\frac{14}{3}}{(2)^2}$$

$$2 \cdot \frac{8}{3} + \left(\frac{14}{3}\cdot\frac{1}{4}\right)$$

$$\frac{14 \div 2}{12 \div 2} \qquad 14 \times 4 = 56 \qquad \frac{7}{6}$$

$$\frac{16}{6} + \frac{7}{6} = \boxed{y_0(2) = \frac{23}{6}}$$

Question #5: If y_1 is the particular solution of the differential equation: $\frac{dy}{dx} - \frac{2y}{x} = 6x^2 - 6$ which satisfies $y(1) = 6$, determine the value of $y_1(2)$.

$$\frac{dy}{dx}(x) - \frac{2}{x}y = (6x^2 - 6)x^{-2} \qquad e^{\int -\frac{2}{x}}$$

$$x^{-2}\left(\frac{dy}{dx} - \frac{2}{x}y\right) = 6 - 6x^{-2} \qquad (x^{-2})$$

$$x^{-2}\frac{dy}{dx} - \cancel{\frac{2x^{-2}y}{x}}$$

$$\frac{1}{-2+1}x^{-2+1} \qquad -x^{-} \qquad -\frac{1}{x}$$

$$\underbrace{x^{-2}}_{f}\underbrace{\frac{dy}{dx}}_{g'} - \underbrace{2x^{-3}}_{f'}\underbrace{y}_{g} = 6 - 6x^{-2}$$

$$\int \frac{d}{dx}(x^{-2}y) = \int 6 - 6x^{-2} \Rightarrow \int 6 - 6\int x^{-2}$$

$$x^{-2}y + \cancel{C} = 6x + \frac{6}{x} + C \qquad 6x + \frac{6}{x} + C$$
$$-\cancel{C} \qquad -C$$

$$\frac{1}{x^{-2}}(x^{-2}y) = \left(6x + \frac{6}{x} + C\right)\frac{1}{x^{-2}}$$

$$y = \frac{6x}{x^{-2}} + \frac{6}{x^{-1}} + \frac{C}{x^{-2}}$$

$$y = 6x^3 + 6x + Cx^2$$

Let $x = 1$, $y = 6$

$$6 = 6(1)^3 + 6(1) + C(1)^2$$

$$6 = 6 + 6 + C$$

$$6 = \cancel{12} + C$$
$$-12 \quad -\cancel{12}$$

$$-6 = C$$

$$y_\phi = 6x^3 + 6x - 6x^2$$

$$y_1(2) = 6(2)^3 + 6(2) - 6(2)^2$$

$$y_1(2) = 48 + 12 - 24$$

$$60 - 24$$

$$\boxed{y_1(2) = 36}$$

Question #6: Solve the differential equation $y' + 2y = 2e^x$.

$$e^{2x}\left(\frac{dx}{dy} + 2y\right) = (2e^x)e^{2x} \qquad e^{\int 2\,dx}$$

$$\underbrace{e^{2x}}_{f}\underbrace{\frac{dx}{dy}}_{g'} + \underbrace{2e^{2x}}_{f'}\underbrace{y}_{g} = 2e^{3x} \qquad e^{2x}$$

$$\int \frac{d}{dx}(e^{2x}y) = \int 2e^{3x}\,dx \Rightarrow 2\int e^{3x}$$

$$2\left(\frac{1}{3}e^{3x}\right)$$

$$e^{2x}y + C = \frac{2}{3}e^{3x} + C$$
$$-C \qquad -C$$

$$\boxed{y = \frac{2}{3}e^x + Ce^{-2x}}$$

Question #7: Solve the differentiation equation

$$\frac{(5+t)\frac{du}{dt} + u = 5 + t}{5(t)t}, \quad t > 0$$

$$\frac{5+t}{}$$

$$e^{\int \frac{1}{5(t)t}\,dt}$$

$$\frac{du}{dt} + \frac{1}{5(t)t}u = 1$$

$$e^{\ln|5(t)+1|}$$

$$\underbrace{5(t)t}_{f}\underbrace{\frac{du}{dt}}_{g'} + \underbrace{1}_{f'}\underbrace{u}_{g} = 5 + t \qquad 5(t) + $$

$$\int \frac{d}{dx}(5(t)t \cdot u) = \int 5(t)t$$

$$5u(t)ut + \cancel{C} = 5t(t)\frac{1}{2}t^2 + C$$
$$-\cancel{C} \qquad -C$$

$$\boxed{u = \frac{5 + (t)\frac{1}{2}t^2 + C}{5(t) + }}$$

$$\boxed{\frac{t^2 + 10t + 2C}{2(t(t)5)}}$$

Question #8: Solve the initial-value problem

$$\frac{t\frac{dy}{dt} + 2y = t^5}{t}, \quad t > 0, \quad y(1) = 0$$

$$\frac{}{t}$$

$$\underbrace{(t)^2}_{f}\underbrace{\frac{dy}{dt}}_{g} + \underbrace{\frac{2}{t}}_{f'}\underbrace{y}_{g'} = t^4(t^2) \qquad e^{\int \frac{2}{t}} \qquad (t^2)$$

$$2\ln|t| \qquad \ln|t^2|$$

$$\int \frac{d}{dx}(t^2 y) = \int t^6$$

$$t^2 y + \cancel{C} = \frac{1}{7}t^7 + C$$
$$-\cancel{C} \qquad -C$$

$$\frac{t^2 y}{t^2} = \frac{\frac{1}{7}t^7 + C}{t^2}$$

$$y = \frac{1}{7}t^5 + \frac{C}{t^2}$$

$$0 = \frac{1}{7}(1)^5 + \frac{C}{(1)^2}$$

$$0 = \frac{1}{7} + C$$

$$-\frac{1}{7} = C$$

$$\boxed{y_0 = \frac{1}{7}t^5 - \frac{1}{7t^2}}$$

$$\boxed{y_0 = \frac{t^5}{7} - \frac{1}{7t^2}}$$