This print-out should have 5 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

If $y = y_0(x)$ is the solution of the differential equation

$$y\frac{dy}{dx} = 4x(64 + y^2),$$

which satisfies $y(0) = 0$, find the value of $y_0(1)$.

1. $y_0(1) = 8\left(e^4 + 1\right)^{1/2}$

2. $y_0(1) = 10\left(e^4 - 1\right)^{1/2}$

3. $y_0(1) = 8e^2$

4. $y_0(1) = 8\left(e^4 - 1\right)^{1/2}$ **correct**

5. $y_0(1) = 9\left(e^4 - 1\right)^{1/2}$

Explanation:

The differential equation

$$y\frac{dy}{dx} = 4x(64 + y^2)$$

becomes

$$\int \frac{2y}{64 + y^2}\,dy = \int 8x\,dx$$

after separating variables and integrating. To evaluate the left hand side we use the substitution $u = 64 + y^2$. Then $du = 2y\,dy$, so

$$\int \frac{1}{u}\,du = \int 8x\,dx.$$

Thus the general solution of this differential equation is given by

$$\ln(64 + y^2) = 4x^2 + C,$$

in which case

$$y = \left(Ae^{4x^2} - 64\right)^{1/2}$$

with A an arbitrary constant. For the particular solution y_0 the value of A is determined by the condition $y(0) = 0$ since

$$y_0(0) = 0 \implies A = 64.$$

Consequently,

$$y_0(x) = 8\left(e^{4x^2} - 1\right)^{1/2},$$

and so

$$\boxed{y_0(1) = 8\left(e^4 - 1\right)^{1/2}.}$$

002 10.0 points

If y_0 satisfies the equations

$$(x^2 + 9)\frac{dy}{dx} = xy, \quad y(0) = 5,$$

determine the value of $y_0(6)$.

1. $y_0(6) = 10^{1/2}\,5$

2. $y_0(6) = 25$

3. $y_0(6) = 30$

4. $y_0(6) = 6 \cdot 5^{1/2}$

5. $y_0(6) = 5^{1/2}\,5$ **correct**

Explanation:

The differential equation

$$(x^2 + 9)\frac{dy}{dx} = xy, \quad y(0) = 5$$

becomes

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 9}\,dx$$

after separating variables and integrating. Thus the general solution of this differential equation is given by

$$\ln y = \frac{1}{2}\ln(x^2+9)+C,$$

which in explicit form is

$$y(x) = A(x^2+9)^{1/2}$$

with A an arbitrary constant. For the particular solution y_0 the condition $y(0) = 5$ determines A uniquely since

$$y(0) = 5 \quad\Longrightarrow\quad A = \frac{5}{3}.$$

Consequently,

$$y_0(x) = \frac{5}{3}(x^2+9)^{1/2},$$

and so at $x = 6$,

$$\boxed{y_0(6) = 5^{1/2}5.}$$

---

003 10.0 points

If y_0 satisfies the equations

$$4\frac{dy}{dx} + \frac{2}{x\,y^3} = 0, \quad y(1) = 2,$$

for x, $y > 0$, find the value of $y_0(e)$.

1. $y_0(e) = 14^{1/4}$ **correct**

2. $y_0(e) = 10^{1/3}$

3. $y_0(e) = 12^{1/4}$

4. $y_0(e) = 6^{1/3}$

5. $y_0(e) = 18^{1/4}$

Explanation:

The differential equation

$$4\frac{dy}{dx} + \frac{2}{x\,y^3} = 0$$

becomes

$$4\int y^3\,dy = -2\int \frac{1}{x}\,dx$$

after separating variables and integrating. Thus the general solution of this differential equation is given by

$$y^4 = -2\ln x + C,$$

which in explicit form can be written as

$$y(x) = (C - 2\ln x)^{1/4}$$

with C an arbitrary constant. For the particular solution y_0 this constant is determined by the condition $y(1) = 2$, for then

$$y(1) = 2 \quad\Longrightarrow\quad 2^4 = C.$$

Hence

$$y_0(x) = (2^4 - 2\ln x)^{1/4},$$

in which case

$$\boxed{y_0(e) = (2^4-2)^{1/4} = 14^{1/4}.}$$

---

004 10.0 points

If $y = y_0(x)$ is the solution of the differential equation

$$\sqrt{9-x^2}\,\frac{dy}{dx} + 3xy = 0$$

which satisfies $y(3) = 6$, find the value of $y_0(0)$.

1. $y_0(0) = -6e^9$

2. $y_0(0) = 6e^9$ **correct**

3. $y_0(0) = -6e^{-9}$

4. $y_0(0) = 6e^{12}$

5. $y_0(0) = 6e^{-9}$

Explanation:

The differential equation

$$\sqrt{9-x^2}\,\frac{dy}{dx} + 3xy \;=\; 0$$

becomes

$$\int \frac{dy}{y} \;=\; -\int \frac{3x}{\sqrt{9-x^2}}\,dx$$

after separating variables and integrating. To evaluate the right hand side we use the substitution $u^2 = 9 - x^2$. Then $u\,du = -x\,dx$, so

$$\ln y \;=\; 3\int du = 3\sqrt{9-x^2} + C.$$

Thus the general solution of the given differential equation is given by

$$y(x) = e^C e^{3\sqrt{9-x^2}} = Ae^{3\sqrt{9-x^2}}$$

with A an arbitrary constant. For the particular solution y_0, the value of A is determined by the condition $y(3) = 6$, since

$$y(3) = 6 \quad\Longrightarrow\quad A = 6.$$

Hence

$$y_0(x) \;=\; 6e^{3\sqrt{9-x^2}},$$

from which it follows that

$$\boxed{y_0(0) \;=\; 6e^9.}$$

---

005 10.0 points

Find the amount A in an account after 5 years when

$$\frac{dA}{dt} \;=\; 0.07A, \quad A(0) \;=\; \$100.$$

1. $A(5) \;=\; \$100e^{3.5}$

2. $A(5) \;=\; \$100e^{-0.35}$

3. $A(5) \;=\; \$100e^{-3.5}$

4. $A(5) \;=\; \$100e^{0.35}$ **correct**

5. $A(5) \;=\; \$100e^{35}$

Explanation:

After integration the differential equation

$$\frac{dA}{dt} \;=\; 0.07A$$

becomes

$$\int \frac{1}{A}\,dA \;=\; \int 0.07dt.$$

Thus

$$\ln A \;=\; 0.07t + C.$$

In turn, after exponentiation this becomes

$$A(t) \;=\; e^C e^{0.07t}$$

with C an arbitrary constant. The constant C is determined by the initial condition $A(0) = 100$ for

$$A(0) = 100 \quad\Longrightarrow\quad e^C \;=\; 100.$$

Consequently, the amount in the account after t years is given by

$$A(t) \;=\; 100e^{0.07t}.$$

At $t = 5$, therefore,

$$\boxed{A(5) \;=\; \$100e^{0.35}}.$$