This print-out should have 7 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Find the slope of the tangent line to the graph of

$$r = e^{\theta} - 8$$

at $\theta = \pi/4$.

1. slope $= \dfrac{1}{e^{\pi/4} - 1}$

2. slope $= \dfrac{1}{4}e^{\pi/4} - 1$ **correct**

3. slope $= e^{\pi/4}$

4. slope $= \dfrac{1}{4}e^{-\pi/4}$

5. slope $= \dfrac{1}{4}e^{\pi/4} + 1$

6. slope $= \dfrac{1}{e^{\pi/4} + 1}$

Explanation:

The graph of a polar curve $r = f(\theta)$ can expressed by the parametric equations

$$x = f(\theta)\cos\theta\,, \quad y = f(\theta)\sin\theta\,.$$

In this form the slope of the polar curve is given by

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)}\,.$$

Now, when

$$r = e^{\theta} - 8\,,$$

we see that

$$y'(\theta) = e^{\theta}\sin\theta + (e^{\theta} - 8)\cos\theta\,,$$

while

$$x'(\theta) = e^{\theta}\cos\theta - (e^{\theta} - 8)\sin\theta\,.$$

But then

$$y'\Big(\frac{\pi}{4}\Big) = \frac{1}{\sqrt{2}}(2e^{\pi/4} - 8)\,, \quad x'\Big(\frac{\pi}{4}\Big) = \frac{8}{\sqrt{2}}\,.$$

Consequently, at $\theta = \pi/4$,

$$\boxed{\text{slope} = \frac{dy}{dx}\Big|_{\theta=\pi/4} = \frac{1}{4}e^{\pi/4} - 1}\,.$$

002 10.0 points

Find the slope of the tangent line to the graph of

$$r = 3 + \sin\theta$$

at $\theta = \pi/6$.

1. slope $= -\dfrac{1}{2}\sqrt{3}$

2. slope $= -\dfrac{1}{3}\sqrt{3}$

3. slope $= -\dfrac{3\sqrt{3}+1}{3+\sqrt{3}}$

4. slope $= -2\sqrt{3}$ **correct**

5. slope $= \dfrac{3\sqrt{3}-1}{\sqrt{3}-3}$

6. slope $= \dfrac{3\sqrt{3}-1}{3+\sqrt{3}}$

Explanation:

The graph of a polar curve $r = f(\theta)$ can expressed by the parametric equations

$$x = f(\theta)\cos\theta\,, \quad y = f(\theta)\sin\theta\,.$$

In this form the slope of the tangent line to the curve is given by

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)}\,.$$

But when

$$r = 3 + \sin\theta\,,$$

we see that

$$y'(\theta) = \cos\theta\sin\theta + (3 + \sin\theta)\cos\theta\,,$$

while

$$x'(\theta) = \cos\theta\cos\theta - (3+\sin\theta)\sin\theta\,.$$

But then

$$y'\Big(\frac{\pi}{6}\Big) = \frac{3\sqrt{3}+1\sqrt{3}}{2}\,,$$

while

$$x'\Big(\frac{\pi}{6}\Big) = \frac{-3}{2}\,.$$

Consequently, at $\theta = \pi/6$,

$$\boxed{\text{slope} = \frac{dy}{dx}\Big|_{\theta=\pi/6} = -2\sqrt{3}}\,.$$

keywords: polar function, slope, parametric equations, tangent line,

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003 10.0 points

Find an equation for the tangent line to the graph of

$$r = 4\cos\theta - 3\sin\theta$$

at $\theta = \pi/4$.

1. $y = \frac{4}{3}x - \frac{1}{8}$

2. $y = \frac{3}{4}x + \frac{1}{4}$

3. $y = \frac{4}{3}x + \frac{1}{4}$

4. $y = \frac{3}{4}x + \frac{1}{8}$ **correct**

5. $y = \frac{3}{4}x - \frac{1}{8}$

6. $y = \frac{4}{3}x + \frac{1}{8}$

Explanation:

The usual point-slope formula can be used to find an equation for the tangent line to the graph of a polar curve $r = f(\theta)$ at a point P once we know the Cartesian coordinates $P(x_0, y_0)$ of P and the slope of the tangent line at P.

Now the graph of $r = f(\theta)$ can expressed by the parametric equations

$$x = f(\theta)\cos\theta\,, \quad y = f(\theta)\sin\theta\,,$$

and in this form the slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\,.$$

In the given case of

$$r = 4\cos\theta - 3\sin\theta\,,$$

therefore,

$$y'(\theta) = 4(\cos^2\theta - \sin^2\theta) - 6\sin\theta\cos\theta\,,$$

while

$$x'(\theta) = -8\sin\theta\cos\theta - 3(\cos^2\theta - \sin\theta)\,.$$

Consequently, the tangent line has

$$\text{slope} = \frac{y'(\pi/4)}{x'(\pi/4)} = \frac{3}{4}\,.$$

On the other hand, at $\theta = \pi/4$,

$$P(x_0, y_0) = \Big(r\cos\frac{\pi}{4},\, r\sin\frac{\pi}{4}\Big) = \Big(\frac{1}{2},\, \frac{1}{2}\Big)\,.$$

Thus, by the point slope formula an equation for the tangent line at P is

$$y - \frac{1}{2} = \frac{3}{4}\Big(x - \frac{1}{2}\Big)$$

which after simplification becomes

$$\boxed{y = \frac{3}{4}x + \frac{1}{8}}\,.$$

keywords: slope, polar graph, parametric equation, tangent line, point slope formula

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004 10.0 points

Find the y-intercept of the tangent line to the graph of

$$r = 3e^{-\theta} - 4$$

at the point P corresponding to $\theta = 0$.

1. y-intercept $= 0$

2. y-intercept $= \frac{2}{3}$

3. y-intercept $= -\frac{1}{3}$

4. y-intercept $= \frac{1}{3}$ **correct**

5. y-intercept $= 1$

Explanation:

The usual point-slope formula can be used to find an equation for the tangent line to the graph of a polar curve $r = f(\theta)$ at a point P once we know the Cartesian coordinates $P(x_0, y_0)$ of P and the slope of the tangent line at P.

Now, when

$$r = 3e^{-\theta} - 4,$$

then

$$x(\theta) = (3e^{-\theta} - 4)\cos\theta,$$

while

$$y(\theta) = (3e^{-\theta} - 4)\sin\theta.$$

Thus in Cartesian coordinates, the point P corresponding to $\theta = 0$ is $(-1, 0)$. On the other hand,

$$x'(\theta) = -3e^{-\theta}\cos\theta - \sin\theta(3e^{-\theta} - 4),$$

while

$$y'(\theta) = -3e^{-\theta}\sin\theta + \cos\theta(3e^{-\theta} - 4),$$

so the slope at P is given by

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{y'(0)}{x'(0)} = \frac{1}{3}.$$

Consequently, by the point slope formula, the tangent line at P has equation

$$y = \frac{1}{3}\Big(x+1\Big),$$

and so has

$$\boxed{y\text{-intercept} = \frac{1}{3}}.$$

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005 10.0 points

Find the length of the curve defined by: $r = \sin\theta$ for , $0 \le \theta \le \pi$.

1. 0

2. 2π

3. π **correct**

4. 4π

5. $\pi/2$

Explanation:

We first find the derivative $\frac{dr}{d\theta} = \cos\theta$. The substitute into the formula for lengh of a curve

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

.

We then have:

$$L = \int_0^\pi \sqrt{\sin^2\theta + \cos^2\theta}\ d\theta = \int_0^\pi 1\ d\theta = \pi$$

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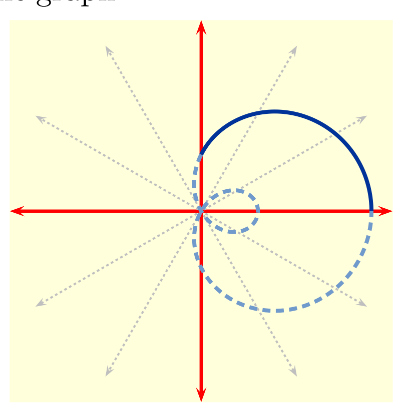
006 10.0 points

Which one of the following integrals gives the arc length of the portion shown as solid blue in the graph



of the polar curve

$$r = 1 + 2\cos\theta.$$

1. $\frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 2\cos\theta)^2\, d\theta$

2. $\frac{1}{2}\int_{0}^{\pi/2} (1 + 2\cos\theta)^2\, d\theta$

3. $\int_{2\pi/3}^{4\pi/3} \sqrt{5 + 4\cos\theta}\, d\theta$

4. $\frac{1}{2}\int_{0}^{2\pi/3} (1 + 2\cos\theta)^2\, d\theta$

5. $\int_{0}^{\pi/2} \sqrt{5 + 4\cos\theta}\, d\theta$ **correct**

6. $\int_{0}^{2\pi/3} \sqrt{5 + 4\cos\theta}\, d\theta$

7. $\frac{1}{2}\int_{\pi/2}^{2\pi} (1 + 2\cos\theta)^2\, d\theta$

8. $\int_{\pi/2}^{2\pi} \sqrt{5 + 4\cos\theta}\, d\theta$

Explanation:

The arc length of a polar curve $r = f(\theta)$ between $\theta = \theta_0$ and $\theta = \theta_1$ is given by the integral

$$L = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + f'(\theta)^2}\, d\theta.$$

Now when

$$r = 1 + 2\cos\theta.$$

its graph will

(i) pass through the origin when $r = 0$, *i.e.*, at $\theta = 2\pi/3,\ 4\pi/3$,

(ii) cross the x-axis also at $\theta = 0,\ \pi$,

(iii) and cross the y-axis also at $\theta = \pi/2,\ 3\pi/2$,

as the graph above indeed shows.

Thus $\theta_0 = 0$ while $\theta_1 = \pi/2$ and

$$\begin{aligned} r^2 + f'(\theta)^2 &= (1 + 2\cos\theta)^2 + 4\sin^2\theta \\ &= 1 + 4\cos\theta + 4\cos^2\theta + 4\sin^2\theta \\ &= 5 + 4\cos\theta. \end{aligned}$$

Consequently

$$\boxed{\text{arc length} = \int_0^{\pi/2} \sqrt{5 + 4\cos\theta}\, d\theta}.$$

keywords: arc length, polar graph, polar curve, trig function, radical, double angle formula

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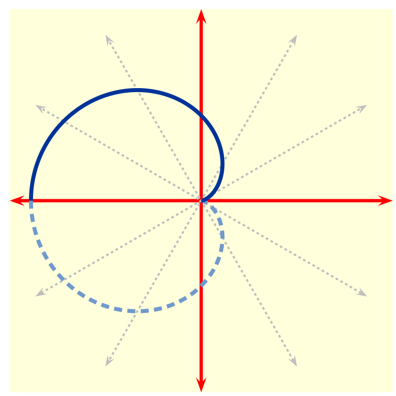
007 10.0 points

Find the arc length of the portion of the graph shown as a solid curve in



of the polar curve

$$r = 1 - \cos\theta.$$

1. arc length $= 2$

2. arc length $= 2 - \sqrt{2}$

3. arc length $= 4$ **correct**

4. arc length $= 2\sqrt{2}$

5. arc length $= 2(2 - \sqrt{2})$

6. arc length $= \sqrt{2}$

Explanation:

The arc length of a polar curve $r = f(\theta)$ between $\theta = \theta_0$ and $\theta = \theta_1$ is given by the integral

$$L = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + f'(\theta)^2}\, d\theta\,.$$

Now in the graph above, $\theta_0 = 0$ while $\theta_1 = \pi$. For

$$r = 1 - \cos\theta\, d\theta$$

the indicated portion of its graph thus has arc length

$$L = \int_0^{\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta\,.$$

But

$$\begin{aligned}(1-\cos\theta)^2 &+ \sin^2\theta \\ &= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta \\ &= 2(1-\cos\theta)\,,\end{aligned}$$

so

$$L = \sqrt{2}\int_0^{\pi} \sqrt{1-\cos\theta}\, d\theta\,.$$

To evaluate this last integral we need to eliminate the radical:

$$\sqrt{1-\cos\theta} = \sqrt{2\sin^2(\theta/2)}\,.$$

Consequently

$$L = 2\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta = -4\Big[\cos\frac{\theta}{2}\Big]_0^{\pi},$$

in which case

$$\boxed{\text{arc length} = L = 4}\,.$$

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keywords: arc length, polar graph, polar curve, trig function, radical, double angle formula