This print-out should have 8 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Determine whether the series

$$\sum_{n=5}^{\infty} \frac{1}{n-3}$$

converges or diverges.

1. series is divergent **correct**

2. series is convergent

Explanation:

We apply the integral test with

$$f(x) = \frac{1}{x-3}.$$

Now f is continuous, positive and decreasing on $[5, \infty)$. Thus the series

$$\sum_{n=5}^{\infty} \frac{1}{n-3}$$

converges if and only if the improper integral

$$\int_5^{\infty} \frac{1}{x-3}\, dx$$

converges. But

$$\int_5^{\infty} \frac{1}{x-3}\, dx = \lim_{t\to\infty} \int_5^t \frac{1}{x-3}\, dx$$

$$= \lim_{t\to\infty} \Big[\ln|x-3|\Big]_5^t$$

$$= \lim_{t\to\infty} \ln\left|\frac{t-3}{5-3}\right| = \infty.$$

Consequently, the

$$\boxed{\text{series } \sum_{n=5}^{\infty} \frac{1}{n-3} \text{ is divergent}}.$$

002 10.0 points

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{3}{n^2+16}$$

converges or diverges.

1. series diverges

2. series converges **correct**

Explanation:

We apply the integral test with

$$f(x) = \frac{3}{x^2+16}.$$

Now f is continuous, positive and decreasing on $[0, \infty)$. Thus the series

$$\sum_{n=1}^{\infty} \frac{3}{n^2+16}$$

converges if and only if the improper integral

$$\int_0^{\infty} \frac{3}{x^2+16}\, dx$$

converges. But

$$\int_0^{\infty} \frac{3}{x^2+16}\, dx = \lim_{t\to\infty} \int_0^t \frac{3}{x^2+16}\, dx.$$

To evaluate this last integral, set $x = 4\tan u$. Then

$$dx = 4\sec^2 u\, du$$

while

$$x = 0 \implies u = 0,$$

$$x = t \implies u = \tan^{-1}\frac{t}{4}.$$

In this case

$$\int_0^t \frac{3}{x^2+16}\, dx = \frac{3}{4}\int_0^{\tan^{-1}(t/4)} \frac{\sec^2 u}{\sec^2 u}\, du$$

$$= \frac{3}{4}\Big[u\Big]_0^{\tan^{-1}(t/4)} = \frac{3}{4}\tan^{-1}\frac{t}{4}.$$

But by properties of $\tan^{-1} t$ we know that

$$\lim_{t\to\infty} \tan^{-1}\frac{t}{4} = \frac{\pi}{2}.$$

Consequently, by the Integral Test, the series

$$\boxed{\sum_{n=1}^{\infty} \frac{3}{n^2+16} \text{ converges}}.$$

003 10.0 points

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{5-3\sqrt{n}}{n^3}$$

converges or diverges.

1. series is convergent **correct**

2. series is divergent

Explanation:

We check separately the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{5}{n^3}, \quad \sum_{n=1}^{\infty} \frac{3\sqrt{n}}{n^3}\left(= \sum_{n=1}^{\infty} \frac{3}{n^{5/2}}\right).$$

Now both are p-series: the first being of the form

$$\sum_{n=1}^{\infty} \frac{5}{n^3} = \sum_{n=1}^{\infty} \frac{5}{n^p}, \quad p=3,$$

while the second one has the form

$$\sum_{n=1}^{\infty} \frac{3}{n^{5/2}} = \sum_{n=1}^{\infty} \frac{3}{n^p}, \quad p = \frac{5}{2}.$$

Since $p > 1$ in both cases, each series converges. As the difference of convergent series, therefore, the given series

$$\boxed{\text{is convergent}}.$$

004 10.0 points

Determine the convergence or divergence of the series

$$(A) \qquad 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots,$$

and

$$(B) \qquad \sum_{m=1}^{\infty} m e^{-m^2}.$$

1. A divergent, B convergent

2. both series divergent

3. both series convergent **correct**

4. A convergent, B divergent

Explanation:

(A) The given series has the form

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is a p-series with $p = 2 > 1$, so the series converges.

(B) The given series has the form

$$\sum_{m=1}^{\infty} f(m)$$

with f defined by

$$f(x) = xe^{-x^2}.$$

Note first that f is continuous and positive on $[1, \infty)$; in addition, since

$$f'(x) = e^{-x^2}(1-2x^2) < 0$$

for $x > 1$, f is decreasing on $[1, \infty)$. Thus we can use the Integral Test. Now, by substitution,

$$\int_1^t xe^{-x^2} dx = \left[-\frac{1}{2}e^{-x^2}\right]_1^t,$$

and so

$$\int_1^{\infty} xe^{-x^2} dx = \frac{1}{2e}.$$

Since the integral converges, the series converges.

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005 10.0 points

Which of the following series are convergent:

A. $$\sum_{n=1}^{\infty} \frac{3}{n^2+1}$$

B. $$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

C. $$\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$$

1. A and B only

2. none of them

3. A only

4. C only

5. all of them **correct**

6. B only

7. B and C only

8. A and C only

Explanation:

By the Integral test, if $f(x)$ is a positive, decreasing function, then the infinite series

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx$$

converges. Thus for the three given series we have to use an appropriate choice of f.

A. Use $f(x) = \dfrac{3}{x^2+1}$. Then

$$\int_1^{\infty} f(x)\,dx$$

is convergent ($\tan^{-1}$ integral).

B. Use $f(x) = \dfrac{1}{x^2}$. Then

$$\int_1^{\infty} f(x)\,dx$$

is convergent.

C. Use $f(x) = \dfrac{2}{x^{3/2}}$. Then

$$\int_1^{\infty} f(x)\,dx$$

is convergent.

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keywords: convergent, Integral test,

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006 10.0 points

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{3}{k(\ln(4k))^2}$$

is convergent or divergent.

1. series converges **correct**

2. series diverges

Explanation:

The function

$$f(x) = \frac{3}{x(\ln(4x))^2}$$

is continous, positive and decreasing on $[1, \infty)$. By the Integral Test, therefore, the series

$$\sum_{k=1}^{\infty} \frac{3}{k(\ln(4k))^2}$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{3}{x(\ln(4x))^2}\,dx$$

converges, *i.e.*, if and only if

$$\lim_{t\to\infty}\int_1^t \frac{3}{x(\ln(4x))^2}\,dx$$

exists. To evaluate this last integral, set $u = \ln(4x)$. Then

$$du = \frac{1}{x}\,dx\,,$$

in which case,

$$\int_1^t \frac{3}{x(\ln(4x))^2}\,dx = \int_{\ln 4}^{\ln(4t)} \frac{3}{u^2}\,du$$

$$= 3\left\{-\frac{1}{\ln(4t)} + \frac{1}{\ln(4)}\right\}.$$

But

$$\lim_{t\to\infty}\frac{1}{\ln(4t)} = 0\,.$$

Consequently,

$$\lim_{t\to\infty}\int_1^t \frac{3}{x(\ln(4x))^2}\,dx$$

exists. The Integral Test thus ensures that the given

series converges.

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007 10.0 points

First find a_n so that

$$\sum_{n=1}^{\infty} a_n = 6 + \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{3}} + \frac{3}{4} + \frac{6}{5\sqrt{5}} + \dots$$

and then determine whether the series converges or diverges.

1. $a_n = \dfrac{6}{n^{1/2}}$, series converges
2. $a_n = \dfrac{6}{n^{1/2}}$, series diverges
3. $a_n = \dfrac{6}{n^{3/2}}$, series converges **correct**
4. $a_n = \dfrac{6}{n^{3/2}}$, series diverges
5. $a_n = \dfrac{3}{2n^{3/2}}$, series converges
6. $a_n = \dfrac{3}{2n^{3/2}}$, series diverges

Explanation:

By inspection

$$\boxed{a_n = \frac{6}{n^{3/2}}}.$$

To test for convergence we use the Integral test with

$$f(x) = \frac{6}{x^{3/2}}\,.$$

This is a positive, continuous, decreasing function on $[1, \infty)$. Furthermore,

$$\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty}\int_1^n \frac{6}{x^{3/2}}\,dx$$

$$= \lim_{n\to\infty}\left[-\frac{12}{x^{1/2}}\right]_1^n = 12\,,$$

so the improper integral

$$\int_1^{\infty} f(x)\,dx$$

is convergent, which by the Integral test means that the

series converges.

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008 10.0 points

Determine whether the series

$$\sum_{m=1}^{\infty} \frac{3\ln(5m)}{m^2}$$

is convergent or divergent.

1. series converges **correct**
2. series diverges

Explanation:

The function

$$f(x) = \frac{3\ln(5x)}{x^2}$$

is continous and positive on $\left[\frac{2}{5}, \infty\right)$; in addition, since

$$f'(x) = 3\left(\frac{1-2\ln(5x)}{x^3}\right) < 0$$

on $\left[\frac{2}{5}, \infty\right)$, f is also decreasing on this interval. This suggests applying the Integral Test, for then the series

$$\sum_{m=1}^{\infty} \frac{3\ln(5m)}{m^2}$$

is convergent if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{3\ln(5x)}{x^2}\,dx$$

converges.

Now, after Integration by Parts, we see that

$$\int_1^t f(x)\,dx = 3\Big[-\frac{\ln(5x)}{x} - \frac{1}{x}\Big]_1^t$$

$$= 3\Big\{-\frac{\ln(5t)}{t} - \frac{1}{t} + \ln(5) + 1\Big\}.$$

Consequently,

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx$$

$$= \lim_{t\to\infty} 3\Big\{-\frac{\ln(5t)}{t} - \frac{1}{t} + \ln(5) + 1\Big\}$$

$$= 3(1+\ln(5)).$$

The Integral Test thus ensures that the given

$$\boxed{\text{series converges}}.$$