Q_1: Determine whether the series

$$\sum_{k=1}^{\infty} \frac{3}{k^7+8}$$

Let $\sum_{k=1}^{\infty} \frac{3}{k^7+8} = \sum a_n$

$\sum b_n = \frac{1}{k^7}$ [converges via p-test]

Limit Comparison

$$\lim_{n\to\infty} \frac{\frac{3}{k^7+8}}{\frac{1}{k^7}} = \lim_{n\to\infty} \frac{3k^7}{k^7+8} \overset{L.H}{=} \lim_{n\to\infty} \frac{21k^{\cancel{6}}}{7k^{\cancel{6}}} = \lim_{n\to\infty} \frac{21}{7} = \lim_{n\to\infty} 3 = \boxed{3}$$

$\boxed{\text{Converges}}$

∴ This series converges via Limit Comparison Test.

Q_2: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$$

Let $\sum a_n = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$

$\sum b_n = \sum \frac{1}{n}$ (diverges by p-test)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cancel{n}(n+1)}{n^{\cancel{2}}} = \lim_{n\to\infty} \frac{n+1}{n} \overset{L.H}{=} \lim_{n\to\infty} \frac{1}{1} = \boxed{1}$$

$\boxed{\text{converges}}$ Limit Comparison Test

Q_3: Which of the following series converges?

a) $\sum_{n=1}^{\infty} \frac{4n}{2n^2+3}$

b) $\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$

c) $\sum_{n=15}^{\infty} \left(\frac{2}{3}\right)^n$

a.) Let $\sum a_n = \sum_{n=1}^{\infty} \frac{4n}{2n^2+3}$

$\sum b_n = \sum \frac{1}{n}$ (diverges via p-test)

Limit Comparison

$$\lim_{n\to\infty} \frac{\frac{4n}{2n^2+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{4n^2}{2n^2+3} \overset{L.H}{=} \lim_{n\to\infty} \frac{8\cancel{n}}{4\cancel{n}} = \lim_{n\to\infty} 2 = 2$$

Converges

This series diverges due to the limit comparison test and the divergences of $\sum b_n$

b) $\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ Let $r = \frac{4}{5}$

If $r \geq 1$ then the series diverges

If $|r| < 1$ then the series converges at some $\frac{a}{1-r}$

This series converges via geom. test

c) $\sum_{n=15}^{\infty} \left(\frac{2}{3}\right)^n$ Let $r = \frac{2}{3}$

This series converges via the geom. test

Q_4: Determine whether the series

$$\sum_{k=1}^{\infty} \frac{2}{4+3^k} \leq \frac{2}{3^k} \text{ (convergent via geom. test)}$$

Let $\sum b_n = \sum_{k=1}^{\infty} \frac{2}{3}\left(\frac{2}{3}\right)^{k-1}$

$\sum_{k=1}^{\infty} \underset{a}{\left(\frac{2}{3}\right)} \underset{r}{\left(\frac{2}{3}\right)}^{k-1}$

and $a_n \leq b_n$ ∴ $\sum a_n$ is $\boxed{\text{Convergent!}}$

$r < 1$ converges

$r \geq 1$ diverges

Q_5: Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)3^k} \qquad \sum \frac{1}{3^k}$$

$\sum b_n = \sum \left(\frac{1}{3}\right)\underset{r}{\left(\frac{1}{3}\right)}^{k-1}$

∴ $\sum a_n$ is convergent

r is less than 1

∴ $\sum b_n$ is convergent via geom

Limit comparison

$$\lim_{k\to\infty} \frac{\frac{k}{(k+1)3^k}}{\frac{1}{3^k}} = \lim_{k\to\infty} \frac{k}{k+1} \overset{LH}{=} \lim_{k\to\infty} \frac{1}{1} = 1$$

$\boxed{\text{converges!}}$

∴ The whole series converges.

Q_7: If a_k, b_k, and c_k satisfy the inequalities

$$0 < a_k \leq c_k \leq b_k,$$

for all k, what can we say about the series

a): $\sum_{k=1}^{\infty} a_k$, b): $\sum_{k=1}^{\infty} b_k$

if we know that the series

c): $\sum_{k=1}^{\infty} c_k$

is divergent but know nothing else about a_k and b_k?

We know $\{a_k, c_k, b_k > 0\}$, which we use DCT to do:

$\{a_n \leq c_k$ and $c_k \leq b_k$

We also know that $\sum c_k$ is divergent

If $a_n \leq c_k$ where c_k is divergent then the test fails and tell us nothing

If $c_k \leq b_n$, where c_k is divergent and due to the conditions of DCT b_k is divergent as well do to the nature of the inequality

Q_7: Determine whether the series

$$\sum_{k=1}^{\infty} \frac{2+k+k^2}{\sqrt{4+k^2+k^6}}$$

The series diverges!

$$\sum_{k=1}^{\infty} \frac{2+k+k^2}{\sqrt{4+k^2+k^6}} = \sum a_n$$

$\sum b_n = \frac{k^2}{k^{6/2}} = \frac{1}{k}$ (diverges via p-test and is a harmonic series).

$$\lim_{n\to\infty} \frac{\frac{2+k+k^2}{\sqrt{4+k^2+k^6}}}{\frac{1}{k}} = \lim_{n\to\infty} \frac{2k+k^2+k^3}{\sqrt{4+k^2+k^6}} = \lim_{n\to\infty} \frac{\frac{2k}{k^3}+\frac{k^2}{k^3}+\frac{k^3}{k^3}}{\sqrt{\frac{4}{k^6}+\frac{k^2}{k^6}+\frac{k^6}{k^6}}}$$

$$= \lim_{n\to\infty} \frac{\frac{2}{k^2}+\frac{1}{k}+1}{\sqrt{\frac{4}{k^6}+\frac{1}{k^3}+1}}$$

$$= \lim_{n\to\infty} \frac{0+0+1}{\sqrt{0+0+1}} = \frac{1}{\sqrt{1}} = \boxed{1} \text{ converges}$$

∴ The series converges via limit comparison test; since $\sum b_n$ diverges, then $\sum a_n$ diverges ∴ this series diverges via limit comparison test.

Q_8: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+2}$$

We know $\cos(n) \leq 1$

∴ $\cos^2(n) \leq 1$

∴ $\frac{\cos^2(n)}{n^2+2} \leq \frac{1}{n^2}$

Since $\frac{\cos^2(n)}{n^2+2} \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges via p-test then:

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+2} \text{ converges.}$$

∴ This whole series converges via direct comparison test. Since $\frac{\cos^2(n)}{n^2+2} \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges via p-test then $\sum \frac{\cos^2(n)}{n^2+2}$ converges.