This print-out should have 8 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{3}{k^7+8}$$

converges or diverges.

1. series is convergent **correct**

2. series is divergent

Explanation:

Note first that the inequalities

$$0 < \frac{3}{k^7+8} < \frac{3}{k^7}$$

hold for all $n \geq 1$. On the other hand, by the p-series test, the series

$$\sum_{k=1}^{\infty} \frac{3}{k^7}$$

is convergent because $p = 7 > 1$. By the comparison test, therefore, the given

$$\boxed{\text{series is convergent}}\,.$$

002 10.0 points

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$$

converges or diverges.

1. series is convergent

2. series is divergent **correct**

Explanation:

Note first that the inequalities

$$\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n} > 0$$

hold for all $n \geq 1$. On the other hand, by the p-series test with $p = 1$ (or by comparison with the harmonic series), the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent. By the comparison test, therefore, the

$$\boxed{\text{series is divergent}}\,.$$

003 10.0 points

Which of the following series

$$(A) \quad \sum_{n=1}^{\infty} \frac{4n}{2n^2+3}$$

$$(B) \quad \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$$

$$(C) \quad \sum_{n=15}^{\infty} \left(\frac{2}{3}\right)^n$$

converge(s)?

1. B only

2. A, B, and C

3. B and C only **correct**

4. C only

5. A and B only

Explanation:

(A) Because of the way the n^{th} term is defined as a quotient of polynomials in the series, use of the integral test is suggested. Set

$$f(x) = \frac{4x}{2x^2+3}\,.$$

Then f is continuous, positive and decreasing on $[1, \infty)$; thus

$$\sum_{n=1}^{\infty} \frac{4n}{2n^2+3}$$

converges if and only if the improper integral

$$\int_1^{\infty} \frac{4x}{2x^2+3}\, dx$$

converges, which requires us to evaluate the integral

$$I_n = \int_1^{n} \frac{4x}{2x^2+3}\, dx\,.$$

Now after substitution (set $u = x^2$), we see that

$$I_n = \Big[\ln(2x^2+3)\Big]_1^n = \Big(\ln(2n^2+3) - \ln(2+3)\Big)\,.$$

But

$$\ln(2n^2+3) \longrightarrow \infty$$

as $n \to \infty$, so the infinite series

$$\sum_{n=1}^{\infty} \frac{4n}{2n^2+3}$$

diverges.

(B) The series

$$\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$$

is a geometric series whose common ratio is $r = \frac{4}{5}$. Since $0 < r < 1$, the series converges; in fact, we know that

$$\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n = \frac{\frac{4}{5}}{1-\frac{4}{5}} = 4$$

because the initial term in the series is $\frac{4}{5}$.

(C) The series

$$\sum_{n=15}^{\infty} \left(\frac{2}{3}\right)^n$$

is a geometric series, but the summmation starts at $n = 15$ instead of at $n = 1$, as it usually does. We can still show, however, that this series converges.

Indeed,

$$\sum_{n=15}^{\infty} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^{15} + \left(\frac{2}{3}\right)^{16} + \left(\frac{2}{3}\right)^{17} + \dots\,.$$

This shows that the series is a geometric series whose common ratio is $r = \frac{2}{3}$ and whose initial term is $(\frac{2}{3})^{15}$. Since $0 < r < 1$, the geometric series converges, and we know that

$$\left(\frac{2}{3}\right)^{15} + \left(\frac{2}{3}\right)^{16} + \left(\frac{2}{3}\right)^{17} + \dots = \frac{\left(\frac{2}{3}\right)^{15}}{1-\frac{2}{3}} = 3\left(\frac{2}{3}\right)^{15}\,.$$

Consequently, of the three infinite series,

$$\boxed{\text{only } B \text{ and } C \text{ converge}}\,.$$

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004 10.0 points

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{2}{4+3^k}$$

converges or diverges.

1. series is divergent

2. series is convergent **correct**

Explanation:

Note first that the inequalities

$$0 < \frac{2}{4+3^k} < \frac{2}{3^k}$$

hold for all $n \geq 1$. On the other hand, the series

$$\sum_{k=1}^{\infty} \frac{2}{3^k}$$

converges because it is a geometric series with

$$|r| = \frac{1}{3} < 1 .$$

By the comparison test, therefore, the

$$\boxed{\text{series is convergent}}\,.$$

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005 10.0 points

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)3^k}$$

converges or diverges.

1. series is divergent

2. series is convergent **correct**

Explanation:

We use the Limit Comparison Test with

$$a_k = \frac{k}{(k+1)3^k}, \quad b_k = \frac{1}{3^k} .$$

For

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k}{k+1} = 1 > 0 .$$

Thus the series

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)3^k}$$

converges if and only if the series

$$\sum_{k=1}^{\infty} \frac{1}{3^k}$$

converges. But this last series is a geometric series with

$$|r| = \frac{1}{3} < 1 ,$$

hence convergent. Consequently, the given series is

$$\boxed{\text{series is convergent}}\,.$$

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006 10.0 points

If a_k , b_k , and c_k satisfy the inequalities

$$0 < a_k \leq c_k \leq b_k ,$$

for all k, what can we say about the series

$$(A) : \sum_{k=1}^{\infty} a_k, \qquad (B) : \sum_{k=1}^{\infty} b_k$$

if we know that the series

$$(C) : \sum_{k=1}^{\infty} c_k$$

is divergent but know nothing else about a_k and b_k?

1. (A) diverges , (B) converges

2. (A) diverges , (B) diverges

3. (A) need not diverge , (B) diverges **correct**

4. (A) converges , (B) need not converge

5. (A) diverges , (B) need not diverge

6. (A) converges , (B) diverges

Explanation:

Let's try applying the Comparison Test:

(i) if

$$0 < a_k \leq c_k, \qquad \sum_k c_k \quad \text{diverges,}$$

then the Comparison Test is inconclusive because $\sum a_k$ could diverge, but it could converge - we can't say precisely without further restrictions on a_k;

(ii) while if

$$0 < c_k \le b_k, \qquad \sum_k c_k \text{ diverges,}$$

then the Comparison Test applies and says that $\sum b_k$ diverges.

Consequently, what we can say is

$$\boxed{(A) \text{ need not diverge}, \quad (B) \text{ diverges}}.$$

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007 10.0 points

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{2+k+k^2}{\sqrt{4+k^2+k^6}}$$

converges or diverges.

1. series is divergent **correct**

2. series is convergent

Explanation:

We apply the Limit Comparison Test with

$$a_k = \frac{2+k+k^2}{\sqrt{4+k^2+k^6}}, \quad b_k = \frac{1}{k}.$$

For then

$$\lim_{k\to\infty} \frac{a_n}{b_n} = \lim_{k\to\infty} \frac{2k+k^2+k^3}{\sqrt{4+k^2+k^6}}$$

$$= \lim_{k\to\infty} \frac{\frac{2}{k^2}+\frac{1}{k}+1}{\sqrt{\frac{4}{k^6}+\frac{1}{k^4}+1}} = 1 > 0.$$

Thus the given series

$$\sum_{k=1}^{\infty} \frac{2+k+k^2}{\sqrt{4+k^2+k^6}}$$

converges if and only if the series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

converges. But, by the p-series test (or because the harmonic series diverges), this last series diverges because $p = 1$. Consequently, the

$$\boxed{\text{series is divergent}}.$$

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008 10.0 points

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+2}$$

converges or diverges.

1. series is divergent

2. series is convergent **correct**

Explanation:

Note first that the inequalities

$$0 < \frac{\cos^2(n)}{n^2+2} \le \frac{1}{n^2+2} \le \frac{1}{n^2}$$

hold for all $n \ge 1$. On the other hand, by the p-series test the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent since $p = 2 > 1$. Thus, by the comparison test, the given

$$\boxed{\text{series is convergent}}.$$