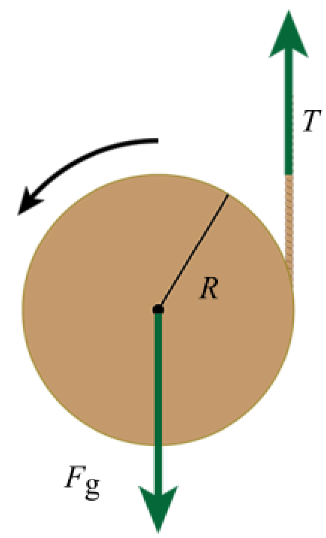
Begin Date: 11/1/2024 12:01:00 AM -- **Due Date:** 11/8/2024 11:59:00 PM **End Date:** 11/8/2024 11:59:00 PM

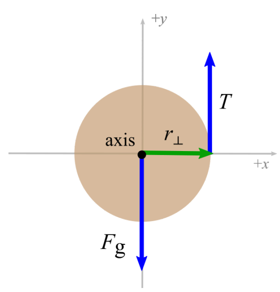
Problem 1:

A rope is wrapped around a solid, uniform disk. The rope is held vertically and the disk is released. The rope unwinds without slipping as the disk falls.



Part (a) **Write an expression for the magnitude of the acceleration of the disk. Your answer will be in terms of g.**

We start with a free body diagram as shown below:



The linear acceleration of the disk will be in the –y-direction. We can apply Newton's second law and solve for an expression for the tension in the rope, T.

$$\begin{aligned} \Sigma F &= -Ma \\ T - F_g &= -Ma \\ T - Mg &= -Ma \\ T &= Mg - Ma \\ &= M(g - a) \end{aligned}$$

Next we apply Newton's second law for rotations, $\Sigma\tau = I\alpha$ and solve for a. To do so, we will need to do the following substitutions.

- The net torque on the disk is due to the tension in the rope acting at the edge of the disk, and can be written as

$$\Sigma\tau = Fr_{\perp} = TR\ .$$

- The moment of inertia of the disk can be obtained from a table. The expression is

$$I = \frac{1}{2}MR^2\ .$$

- For the angular acceleration, α, we can use the relation $a = R\alpha$ and rearrange as

$$\alpha = \frac{a}{R}\ .$$

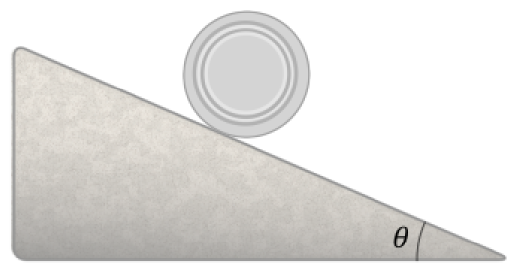
Substituting in for T and carrying out the calulation:

$$\Sigma\tau = I\alpha$$
$$TR = \left(\frac{1}{2}MR^2\right)\left(\frac{a}{R}\right)$$
$$\cancel{M}(g - a)\cancel{R} = \frac{1}{2}\cancel{M}\cancel{R}a$$
$$g - a = \frac{1}{2}a$$
$$\frac{3}{2}a = g$$
$$a = \frac{2}{3}g$$

$$\boxed{a = \frac{2}{3}g}$$

Problem 2:

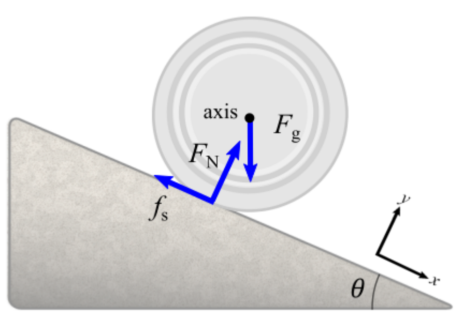
A can of beans rolls without slipping down an incline that makes and angle of θ with horizontal. It is released from rest. Assume that the can is a uniform solid cylinder.



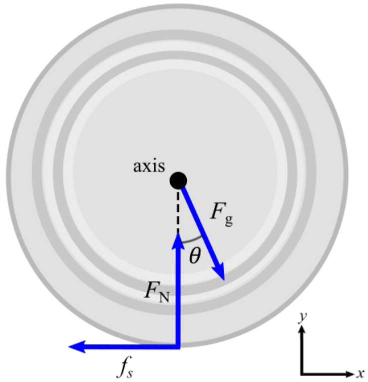
Part (a) **Determine an expression for the linear acceleration of the can. Your answer will be in terms of θ and g.**

To solve, we apply Newton's second law for both linear and rotational motion.

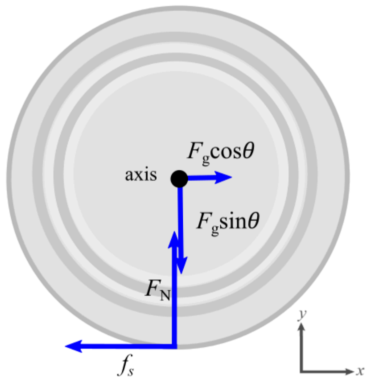
The forces on the can are shown in the following diagram:



We can rotate the disk through an angle $-\theta$,



and break the forces up into their components.



The can rolls because because of the torque created by the frictional force. Because the lines of action of the gravitational force and the normal force cut through the axis of rotation, they cannot contribute to the total torque on the can. If we take R to be the radius of the can, we can express the magnitude of the torque from the frictional force as

$$\tau = f_s R \sin\theta = f_s R\ .$$

Next we apply Newton's second law for rotations with a few substitutions. The moment of inertia of a solid cylinder is $I = \frac{1}{2}MR^2$, and can be obtained from a table. Since we are looking for the linear acceleration, a, it will also be useful to make a substitution involving $\alpha = \frac{a}{R}$.

$$\Sigma\tau = I\alpha$$
$$f_s \cancel{R} = \frac{1}{2}M\cancel{R}^2\left(\frac{a}{\cancel{R}}\right)$$
$$f_s = \frac{1}{2}Ma$$

Next we will apply Newton's second law in the x-direction to solve for a. We will need to substitute the expression for f_s from above into our calculation.

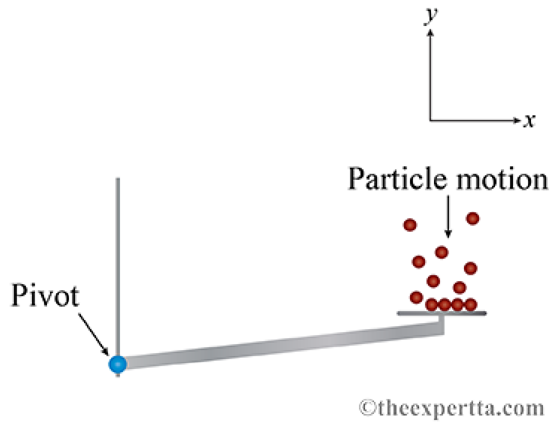
$$\Sigma F_x = Ma$$
$$F_g \sin\theta - f_s = Ma$$
$$F_g \sin\theta - \frac{1}{2}Ma = Ma$$
$$\frac{3}{2}\cancel{M}a = \cancel{M}g\sin\theta$$
$$a = \frac{2}{3}g\sin\theta$$

$$\boxed{a = \frac{2}{3}g\sin\theta}$$

Problem 3:

Gallium atoms of mass $m = 1.16 \times 10^{-25}$ kg are adsorbed from a vapor to a small horizontal pan attached to a microscopic beam. The beam is attached to a post by a pivot and is inclined above the horizontal, as shown in the figure. The weight of the particles that the pan adsorbs exerts a force F on the end of the beam. The beam was designed so that it does not rotate until the torque about the pivot due to the particles reaches $\tau = 1.02 \times 10^{-20}$ N · m.

The vector extending along the beam from the pivot to the point at which the weight of the particles acts on the beam is $\vec{r} = x\,\hat{i} + y\,\hat{j}$ in the coordinate system shown in the figure.



Part (a) **What is the direction of the torque that the atoms produce on the beam?**

We can find the direction of the torque by using the right hand rule. If we imagine the beam to be a typical screw with the head facing us, we can see that the clockwise rotation of the beam caused by the weight of the atoms would move the screw's head into the page, which is the negative z direction.

The negative z-direction.

Part (b) **If $x = 2.01 \times 10^{-6}$ m, and $y = 0.25 \times 10^{-6}$ m, then what is the minimum number of gallium atoms necessary to rotate the beam?**

To create a torque about a designated pivot point, a force must act at least somewhat perpendicularly to a line connecting that point to the point where the force is applied. In this case, that line points from the pivot to the pan. Only the perpendicular part of that line contributes to torque (that is, x). The torque is then given by

$$\tau = r \times F = Wx$$

where W is the collective weight of the gallium atoms. For rotation to just begin, the torque equals the number provided in the problem statement. Substitution yields

$$1.02 \cdot 10^{-20} = W\left(2.01 \times 10^{-6}\right)$$

$$W = 0.5075 \times 10^{-14}\ \text{N}$$

We are told that the mass of each atom is

$$m = 1.16 \times 10^{-25}\ \text{kg}$$

and so the weight of the *n* atoms that have accumulated on the pan is

$$W = n\,(9.81)\left(1.16 \times 10^{-25}\right)\ \text{N}$$

Setting the two expressions equal to each other allows us to solve for *n*

$$0.5075 \times 10^{-14}\ \text{N} = n\,(9.81)\left(1.16 \times 10^{-25}\right)\ \text{N}$$

$$n = 0.04459 \times 10^{11}$$

$$\boxed{n = 4.459 \times 10^{9}}$$

Part (c) **If the beam were initially oriented at a greater angle above the horizontal, how would your answer in Part (b) change?**

To create a torque about a designated pivot point, a force must act at least somewhat perpendicularly to a line connecting that point to the point where the force is applied. In this case, that line points from the pivot to the pan. Only the perpendicular part of that line contributes to torque (that is, x). The torque is then given by

$$\tau = r \times F = xW$$

where W is the collective weight of the gallium atoms. For rotation to just begin, the torque equals the number provided in the problem statement. Substitution yields

$$1.02 \cdot 10^{-20} = xW$$

If the same beam is at a greater angle above the horizontal, then the y component of the position vector from pivot to pan increases, but the x component decreases. If x decreases, then W must increase. That means that more atoms must accumulate.

The number of atoms must increase

Problem 4:

A force $\vec{F} = (2.01\,\hat{i} + (-4.99)\,\hat{j} + 1.01\,\hat{k})$ N is applied at a point whose position is $\vec{r} = (-3.99\,\hat{i} + 2.01\,\hat{j})$ m.

Part (a) **What is the magnitude of the torque, in units of newton meters, generated by this force about the origin?**

When given the force and position vectors in terms of unit vectors, the torque can be determined by means of the cross product. This can be written as a determinant:

$$\begin{aligned}\vec{\tau} &= r \times F \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \\ &= (r_yF_z - r_zF_y)\hat{i} - (r_xF_z - r_zF_x)\hat{j} + (r_xF_y - r_yF_x)\hat{j} \\ &= [(2.01\text{ m})(1.01\text{ N}) - (0)(-4.99)\text{ N}]\hat{i} - [(-3.99\text{ m})(1.01\text{ N}) - (0)(2.01\text{ N})]\hat{j} + [(-3.99\text{ m})(-4.99\text{ N}) - (2.01\text{ m})(2.01\text{ N})]\hat{k} \\ &= (2.030\text{ N}\cdot\text{m})\hat{i} - (-4.030\text{ N}\cdot\text{m})\hat{j} + (15.87\text{ N}\cdot\text{m})\hat{k}\end{aligned}$$

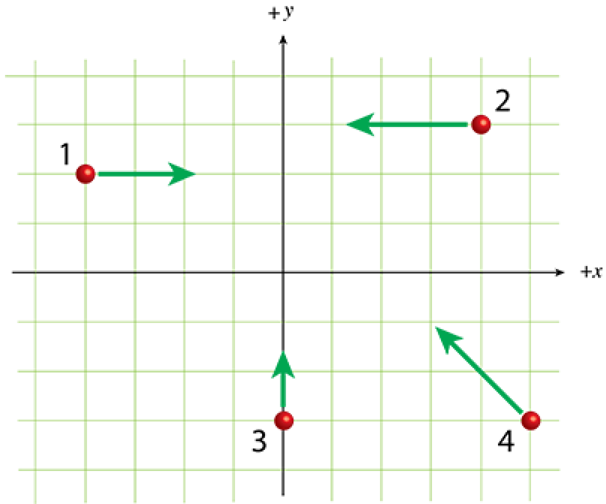
To determine the magnitude of the torque, we apply Pythagorean theorem.

$$\begin{aligned}\tau &= \sqrt{(2.030\text{ N}\cdot\text{m})^2 + (-4.030\text{ N}\cdot\text{m})^2 + (15.87\text{ N}\cdot\text{m})^2} \\ &= 16.50\text{ N}\cdot\text{m}\end{aligned}$$

$\tau = 16.50\text{ N}\cdot\text{m}$

Problem 5:

The diagram shows four particles (labeled 1, 2, 3, and 4) that are moving in the *x*-*y* plane. Particle 1 is at (-4 m, 2 m) and is moving in the positive *x* direction. Particle 2 is at (4 m, 3 m) and traveling in the negative *x* direction. Particle 3 is at (0 m, -3 m) and moving in the positive *y* direction. Particle 4 is at (5 m, -3 m) and its velocity makes an angle of 45 degrees with the coordinate axes.



Part (a) **What is the direction of the angular momentum of particle 1 about the origin?**

We learn from our textbook that when the motion in the x-y plane is counterclockwise/clockwise with respect to the origin, the angular momentum about the origin is in the positive/negative z direction.

Particle 1 is moving clockwise, so the correct choice is ...

Negative z

Part (b) **What is the direction of the angular momentum of particle 2 about the origin?**

We learn from our textbook that when the motion in the x-y plane is counterclockwise/clockwise with respect to the origin, the angular momentum about the origin is in the positive/negative z direction.

Particle 2 is moving counterclockwise, so the correct choice is ...

Positive z

Part (c) **What is the direction of the angular momentum of particle 3 about the origin?**

We learn from our textbook that when the motion in the x-y plane is through the origin, the angular momentum about the origin is zero.

Particle 3 is moving through the origin, so the correct choice is ...

The angular momentum is zero.

Part (d) **What is the direction of the angular momentum of particle 4 about the origin?**

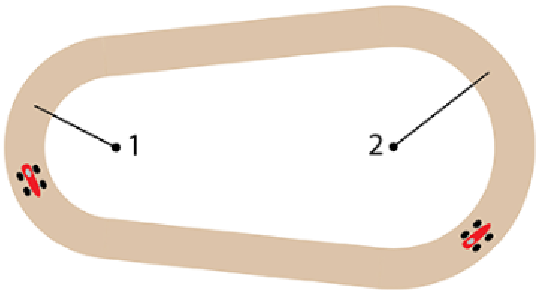
We learn from our textbook that when the motion in the x-y plane is counterclockwise/clockwise with respect to the origin, the angular momentum about the origin is in the positive/negative z direction.

Particle 4 is moving counterclockwise, so the correct choice is ...

Positive z

Problem 6:

A Formula One race car drives at constant speed around a track that has two semicircular turns of radii r_1 and r_2. Let l_1 be the angular momentum of the race car about the center of turn 1 when it is driving along turn 1, and l_2 be the angular momentum of the race car about the center of turn 2 when it is driving along turn 2.



Part (a) **Write an expression for the ratio $\frac{l_2}{l_1}$.**

Here we treat the Formula One car like a particle. The angular momentum $\vec{l}$ is given by

$$l = \vec{r} \times \vec{p},$$

where $\vec{r}$ is the position of the particle with respect to a reference point, and $\vec{p} = m\vec{v}$ is the particle's linear momentum. The magnitude can be expressed as $l = mvr \sin\theta$, However, we only know the length of vector r when it is equal to the turning radius. This is a convenient spot to determine the angular momentum because when the car is at this location, $\vec{r} \perp \vec{p}$, so $\sin\theta = 90^{\circ}$. This makes the angular momentum with respect to the small center of curvature

$$l_1 = mvr_1 \ ,$$

and the angular momentum with respect to the large center of curvature

$$l_2 = mvr_2 \ .$$

The ratio is therefore

$$\frac{l_2}{l_1} = \frac{\cancel{m}\cancel{v}r_2}{\cancel{m}\cancel{v}r_1} = \frac{r_2}{r_1}$$

$$\boxed{\frac{l_2}{l_1} = \frac{r_2}{r_1}}$$

Part (b) **If** $r_1 = 31$ m **and** $r_2 = 71$ m **what is the numerical value of** $\frac{l_2}{l_1}$**?**

It can be shown that the ratio of the angular momenta is

$$\frac{l_2}{l_1} = \frac{r_2}{r_1} \ .$$

Plugging in:

$$\frac{l_2}{l_1} = \frac{r_2}{r_1} = \frac{71 \text{ m}}{31 \text{ m}} = 2.290$$

$$\boxed{\frac{l_2}{l_1} = 2.290}$$

Problem 7:

Suppose you start an antique car by exerting a force of *250* N on its crank for *0.13* s.

Randomized Variables

$f = 250$ N
$t = 0.13$ s
$d = 0.22$ m

Part (a) **What angular momentum is given to the engine if the handle of the crank is *0.22* m from the pivot and the force is exerted to create maximum torque the entire time?**

Recall that torque is analogous to force. Therefore, just as the force multiplied by time (the impulse) is equal to the change in momentum, so too is the torque multiplied by time equal to the change angular momentum. Let's write an equation for this.

$$\tau t = \Delta L$$

$$\tau t = L - L_0$$

$$\tau t = L - 0$$

$$\tau t = L$$

Since the force is being applied in such a way as to maximize the torque, we know that the force must be being applied directly perpendicular to the lever arm. With this in mind, we can write the following equation for the torque.

$$\tau = Fr\sin(90^\circ)$$

$$\tau = Fr$$

$$\tau = Fd$$

Using this expression, we can now solve our previous equation for the angular momentum.

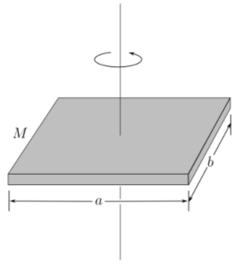
$$Fdt = L$$

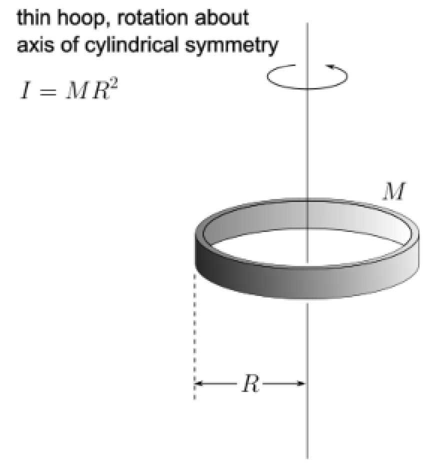
$$L = 250\text{ N} \cdot 0.22\text{ m} \cdot 0.13\text{ s}$$

$$\boxed{L = 7.150\text{ kg} \cdot \text{m}^2/\text{s}}$$

Problem 8:

A solid slab with a mass of 2.01 kg, a width of 10.2 cm, and a length of 25.3 cm is free to rotate about the z axis, which passes through its center and is perpendicular to its face. Initially at rest, a constant torque, $\vec{\tau} = (12.5 \text{ mN} \cdot \text{m})\,\hat{k}$, is applied through 5.24 s.





In lieu of the typical table as found in most textbooks, the MOIs of uniform standard objects may be obtained by using the left and right arrows to step through the image carousel to the right.

Part (a) **What is the magnitude, in millijoule seconds, of the final angular momentum of the slab?**

With linear variables, the ordinary impulse is related to the change in momentum as

$$d\vec{p} = \vec{F}\, dt$$

With angular variables, the angular impulse is related to the change in the angular momentum as

$$d\vec{L} = \vec{\tau}\, dt$$

When the torque is constant, the relationship is exact for finite differences.

$$\Delta\vec{L} = \vec{\tau}\, \Delta t$$

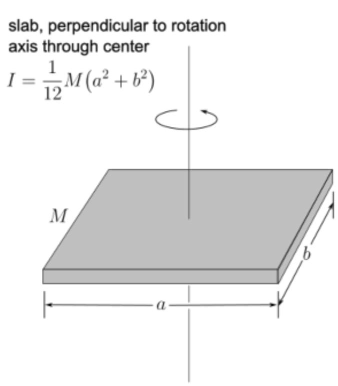
In the case at hand, both the torque and the change in the angular momentum will be proportional to the $\hat{k}$ unit vector. Retaining only the scalar coefficients, and noting that the slab starts from rest,

$$\begin{aligned} L_f &= L_0 + \tau\,\Delta t \\ &= 0 + (12.5 \times 10^{-3} \text{ N} \cdot \text{m})(5.24 \text{ s}) \\ &= 0.06550 \text{ J} \cdot \text{s} \times \frac{1000 \text{ mJ}}{1 \text{ J}} \end{aligned}$$

$$\boxed{L_f = 65.50 \text{ mJ} \cdot \text{s}}$$

Part (b) **What is the final angular velocity, in revolutions per minute, after the application of the torque?**

Obtain the formula for the MOI from the image carousel in the problem statement. For the slab rotating about an axis through its center and perpendicular to its face, use the following template:



The axes coincide, so the Parallel Axis Theorem is not needed. Using the correct parameters for the current context,

$$
\begin{aligned}
I &= \frac{1}{12}M\left(a^2+b^2\right) \\
&= \frac{1}{12}(2.01\ \text{kg})\left[\left(10.2\times10^{-2}\ \text{m}\right)^2+\left(25.3\times10^{-2}\ \text{m}\right)^2\right] \\
&= 0.01246\ \text{kg}\cdot\text{m}^2
\end{aligned}
$$

Angular momentum is proportional to angular velocity, and the MOI is the constant of proportionality.

$$L = I\omega$$

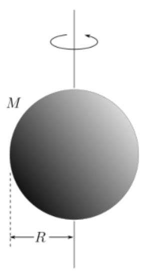
Solving for the angular velocity and making substitutions,

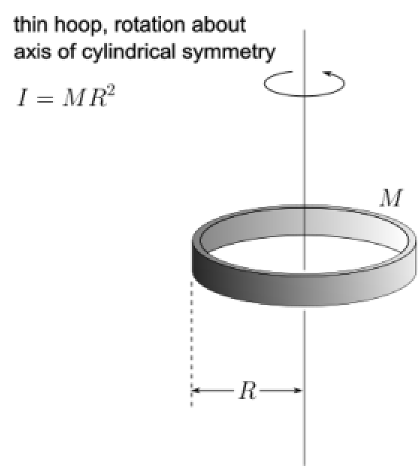
$$
\begin{aligned}
\omega &= \frac{L}{I} \\
&= \frac{65.50\times10^{-3}\ \text{J}\cdot\text{s}}{0.01246\ \text{kg}\cdot\text{m}^2} \\
&= 5.255\ \text{rad/s}\times\frac{1\ \text{rev}}{2\pi\ \text{rad}}\times\frac{60\ \text{s}}{1\ \text{m}}
\end{aligned}
$$

$\omega_f = 50.18$ rpm

Problem 9:

A solid sphere with a mass of 2.01 kg and a radius of 25.2 cm is free to rotate about the z axis, which passes through its center. Initially at rest, a constant torque, $\vec{\tau} = (12.5\ \text{mN}\cdot\text{m})\,\hat{k}$, is applied through 5.14 s.





In lieu of the typical table as found in most textbooks, the MOIs of uniform standard objects may be obtained by using the left and right arrows to step through the image carousel to the right.

Part (a) **What is the magnitude, in millijoule seconds, of the final angular momentum of the solid sphere?**

With linear variables, the ordinary impulse is related to the change in momentum as

$$d\vec{p} = \vec{F}\,dt$$

With angular variables, the angular impulse is related to the change in the angular momentum as

$$d\vec{L} = \vec{\tau}\,dt$$

When the torque is constant, the relationship is exact for finite differences.

$$\Delta\vec{L} = \vec{\tau}\,\Delta t$$

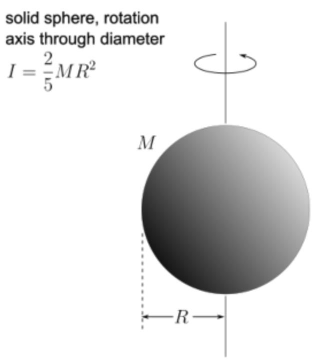
In the case at hand, both the torque and the change in the angular momentum will be proportional to the $\hat{k}$ unit vector. Retaining only the scalar coefficients, and noting that the solid sphere starts from rest,

$$
\begin{aligned}
L_f &= L_0 + \tau\,\Delta t \\
&= 0 + \left(12.5\times10^{-3}\ \text{N}\cdot\text{m}\right)(5.14\ \text{s}) \\
&= 0.06425\ \text{J}\cdot\text{s}\times\frac{1000\ \text{mJ}}{1\ \text{J}}
\end{aligned}
$$

$$L_f = 64.25 \text{ mJ} \cdot \text{s}$$

Part (b) **What is the final angular velocity, in revolutions per minute, after the application of the torque?**

Obtain the formula for the MOI from the image carousel in the problem statement. For the solid sphere rotating about an axis through its center, use the following template:



The axes coincide, so the Parallel Axis Theorem is not needed. Using the correct parameters for the current context,

$$\begin{aligned} I &= \frac{2}{5}MR^2 \\ &= \frac{2}{5}(2.01 \text{ kg})(25.2 \times 10^{-2} \text{ m})^2 \\ &= 0.05106 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

Angular momentum is proportional to angular velocity, and the MOI is the constant of proportionality.

$$L = I\omega$$

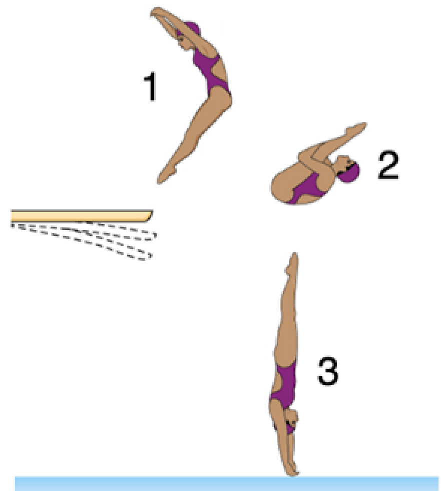
Solving for the angular velocity and making substitutions,

$$\begin{aligned} \omega &= \frac{L}{I} \\ &= \frac{64.25 \times 10^{-3} \text{ J} \cdot \text{s}}{0.05106 \text{ kg} \cdot \text{m}^2} \\ &= 1.258 \text{ rad/s} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} \times \frac{60 \text{ s}}{1 \text{ m}} \end{aligned}$$

$$\omega_f = 12.02 \text{ rpm}$$

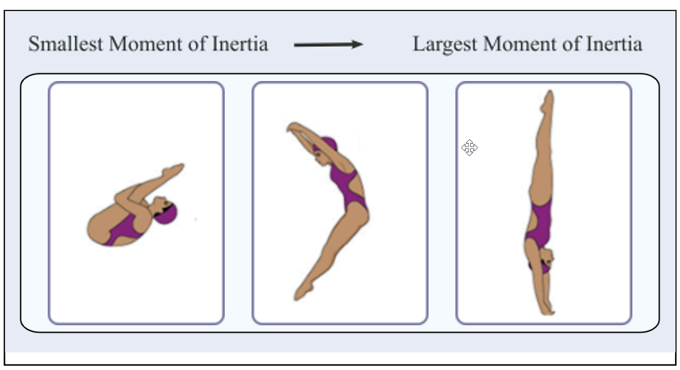
Problem 10:

A diver does a backflip off a 3-meter board, as shown in the diagram. Assume the diver's angular momentum is conserved as she moves from position 1 to 2 to 3. At position 1 she is beginning to tuck her body, at position 2 she is in a full tuck, and just before entering the water at position 3 she is fully extended.



Part (a) **Drag and drop the images of the diver to the space below and order them from her smallest to largest moment of inertia. .**

Moment of inertia depends on how the mass is distributed about an axis of rotation. In general, objects with most of the mass close to the axis of rotation have relatively small moments of inertia, and objects with most of the mass far away from the axis of rotation have relatively large moments of inertia. In this case, the diver has the greatest moment of inertia when she is stretched out straight, and the smallest moment of inertia when she is fully tucked.

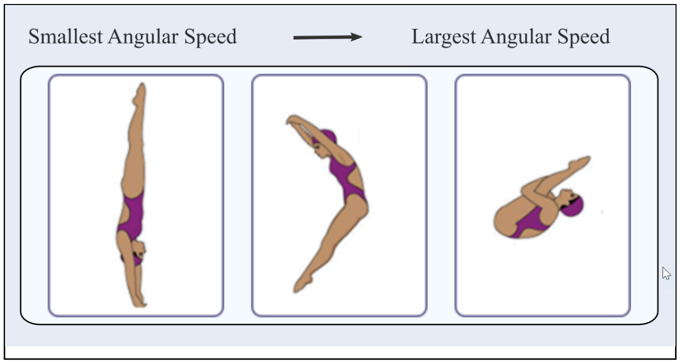


Part (b) **Drag and drop the images of the diver to the space below and order them from her smallest to largest angular speed.**

According to conservation of angular momentum, the diver's angular momentum is constant throughout the dive. In other words,

$$L = I\omega = \text{constant} .$$

Thus, as her moment of inertia changes during the dive, her angular speed must change as well. For angular momentum to remain constant, her angular speed must be small when her moment of inertia is large and vice versa.



Problem 11:

Consider a bike wheel that is hanging from a rope tied to one end of its axle (see figure).

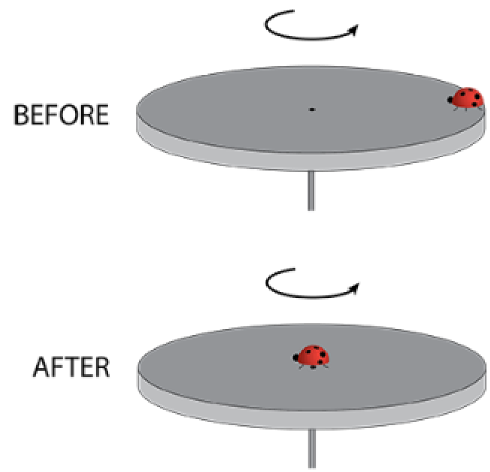
Part (a) **If you get the bike wheel spinning quickly about its axis and let it go, what direction will it move?**

The spinning bicycle wheel has a large angular momentum that points along the axis of rotation for the wheel. The force of gravity acts on the wheel in a way that produces a torque that is perpendicular to the angular momentum. This torque causes the wheel to precess, or rotate,...

Around the vertical axis that coincides with the rope.

Problem 12:

A bug of mass m is at the edge of a horizontal disk of mass M and radius R that rotates freely about an axis through its center with an angular speed of ω_i. Starting at the edge of the disk, the bug walks towards its center.



Part (a) **Write an expression for the angular speed ω_f of the disk when the bug is at the center. Your answer will be in terms of M, m, and ω_i.**

Since there are no external torques on the system, conservation of angular momentum can be applied. To do so, we need the initial and final moments of inertia of the system. The bug can be treated as a particle a distance R from the axis of rotation. This makes the initial moment of inertia

$$I_i = I_{disk} + I_{bug} = \frac{1}{2}MR^2 + mR^2 .$$

After the bug walks to the center of the disk, its distance to the axis of rotation is zero, so it no longer contributes to the total moment of inertia. The final moment of inertia becomes

$$I_f = I_{disk} = \frac{1}{2}MR^2 .$$

Applying conservation of angular momentum:

$$\begin{aligned}
L_f &= L_i \\
I_f\omega_f &= I_i\omega_i \\
\left(\frac{1}{2}MR^2\right)\omega_f &= \left(\frac{1}{2}MR^2 + mR^2\right)\omega_i \\
\omega_f &= \frac{\frac{1}{2}MR^2 + mR^2}{\frac{1}{2}MR^2}\omega_i \\
&= \frac{(M+2m)\cancel{R^2}}{M\cancel{R^2}}\omega_i \\
&= \frac{M+2m}{M}\omega_i \\
&= \left(1 + \frac{2m}{M}\right)\omega_i
\end{aligned}$$

$$\boxed{\omega_f = \left(1 + \frac{2m}{M}\right)\omega_i}$$

Part (b) **Write an expression for the change (final minus initial) in kinetic energy of the system as the bug crawls to the center of the disk. Your answer will be in terms of R, M, m, and ω_i.**

In this problem the axis is fixed, so all of the kinetic energy is rotational kinetic energy. In general, the kinetic energy of a rotating body can be expressed as

$$K_R = \frac{1}{2}I\omega^2 .$$

If the final angular speed is equal to

$$\omega_f = \left(1 + \frac{2m}{M}\right)\omega_i ,$$

and the initial and final moments of inertia are

$$I_i = \frac{1}{2}MR^2 + mR^2$$

and

$$I_f = \frac{1}{2}MR^2 ,$$

the change in the rotational kinetic energy is the following:

$$\begin{aligned}
\Delta K &= K_{R,f} - K_{R,i} \\
&= \frac{1}{2}I_f\omega_f^2 - \frac{1}{2}I_i\omega_i^2 \\
&= \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(1 + \frac{2m}{M}\right)^2\omega_i^2 - \frac{1}{2}\left(\frac{1}{2}MR^2 + mR^2\right)\omega_i^2 \\
&= \frac{1}{4}MR^2\left(1 + \frac{4m}{M} + \frac{4m^2}{M^2}\right)\omega_i^2 - \left(\frac{1}{4}MR^2 + \frac{1}{2}mR^2\right)\omega_i^2 \\
&= \left(\frac{1}{4}M + m + \frac{m^2}{M} - \frac{1}{4}M - \frac{1}{2}m\right)R^2\omega_i^2 \\
&= \left(\frac{m}{2} + \frac{m^2}{M}\right)R^2\omega_i^2 \\
&= \left(\frac{1}{2} + \frac{m}{M}\right)mR^2\omega_i^2
\end{aligned}$$

$$\boxed{\Delta K = \left(\frac{1}{2} + \frac{m}{M}\right)mR^2\omega_i^2}$$

Problem 13:

The Earth spins on its axis and also orbits around the Sun. For this problem use the following constants.

Mass of the Earth: 5.97×10^{24} kg (assume a uniform mass distribution)
Radius of the Earth: 6371 km
Distance of Earth from Sun: 149,600,000 km

Part (a) **Calculate the rotational kinetic energy of the Earth due to rotation about its axis, in joules.**

The rotational kinetic energy of the Earth is given by

$$K_r = \frac{1}{2}I\omega^2$$

where

$$I = \frac{2}{5}MR_E^2$$

and

$$\omega = \left(\frac{1\ \text{rev}}{24\ \text{h}}\right)\left(2\pi\frac{\text{rad}}{1\ \text{rev}}\right)\left(\frac{1\ \text{h}}{3600\ \text{s}}\right) = 7.272 \times 10^{-5}\ \text{rad/s}$$

$$K_r = \frac{1}{2}\left(\frac{2}{5}\right) MR_E^2 \omega^2$$

$$= \frac{1}{5}\left(5.97 \times 10^{24}\ \text{kg}\right)\left(6.371 \times 10^{6}\ \text{m}\right)^2\left(7.272 \times 10^{-5}\ \text{rad/s}\right)^2$$

$$\boxed{K_r = 2.563 \times 10^{29\ \text{J}}}$$

Part (b) **What is the rotational kinetic energy of the Earth due to its orbit around the Sun, in joules?**

The kinetic energy of the Earth as it orbits the Sun is given by

$$K_{r,Sun} = \frac{1}{2} I \omega^2$$

At a massive distance like this, we can approximate the Earth as a point mass so the moment of inertia is

$$I = MR_{\text{Earth-Sun}}^2$$

and

$$\omega = \left(\frac{1\ \text{rev}}{1\ \text{y}}\right)\left(2\pi\frac{\text{rad}}{1\ \text{rev}}\right)\left(\frac{1\ \text{y}}{365\ \text{days}}\right)\left(\frac{1\ \text{day}}{24\ \text{h}}\right)\left(\frac{1\ \text{h}}{3600\ \text{s}}\right) = 1.992 \times 10^{-7}\ \text{rad/s}$$

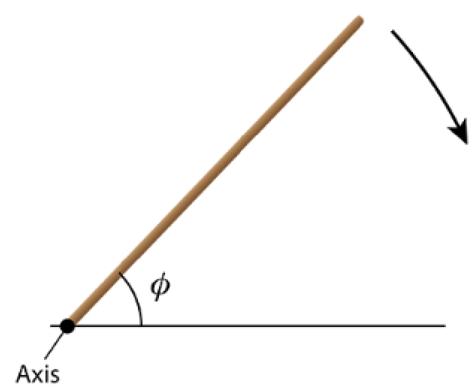
$$K_{r,Sun}\, r = \frac{1}{2} MR_{\text{Earth-Sun}}^2 \omega^2$$

$$= \frac{1}{2}\left(5.97 \times 10^{24}\ \text{kg}\right)\left(1.496 \times 10^{11}\ \text{m}\right)^2\left(1.992 \times 10^{-7}\ \text{rad/s}\right)^2$$

$$\boxed{K_{r,Sun} = 2.65 \times 10^{33}\ \text{J}}$$

Problem 14:

A uniform rod with length L is free to rotate about an axis at ground level. The rod makes an angle ϕ with horizontal and is released from rest.

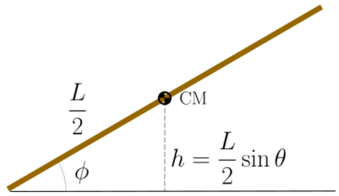


Part (a) **Write an expression for the angular speed of the rod just before it strikes the ground.**

This problem involves conservation of total mechanical energy. Initially, all of the total mechanical energy is the potential energy of the rod in the raised position. Just before the rod strikes the ground, all of the potential energy has been converted to rotational kinetic energy. This reduces the conservation of total mechanical energy equation in the following way.

$$E_i = E_f$$
$$K_i + U_i + K_{R\,i} = K_f + U_f + K_{R\,f}$$
$$U_i = K_{R\,f}$$

To determine an expression for the potential energy, we must consider the height of the center of mass of the rod. This can be determined using trigonometry, as shown in the image below.



The height of the center of mass of the rod is

$$h = \frac{L}{2}\sin\phi\ .$$

The rotational kinetic energy of the rod just before it strikes the ground. It is given by

$$K_{R\,f} = \frac{1}{2}I\omega^2\ ,$$

where the moment of inertia of a rod rotating about one end is $I = \frac{1}{3}mL^2$.

Applying conservation of energy:

$$K_{R\,f} = U_i$$
$$\frac{1}{2}I\omega^2 = mgh$$
$$\frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega^2 = mg\frac{L}{2}\sin\phi$$
$$\omega^2 = \frac{3g\sin\phi}{L}$$
$$\omega = \sqrt{\frac{3g\sin\phi}{L}}$$

$$\boxed{\omega = \sqrt{\frac{3g\sin\phi}{L}}}$$

Part (b) **If the rod has a length of 1.5 m and is dropped from an angle of 20.1° above the horizontal, what is the linear speed of the tip of the rod, in meters per second, just before the rod strikes the ground?**

Using conservation of energy, it can be shown that the angular speed of the rod just before it strikes the ground is

$$\omega = \sqrt{\frac{3g\sin\phi}{L}}\ .$$

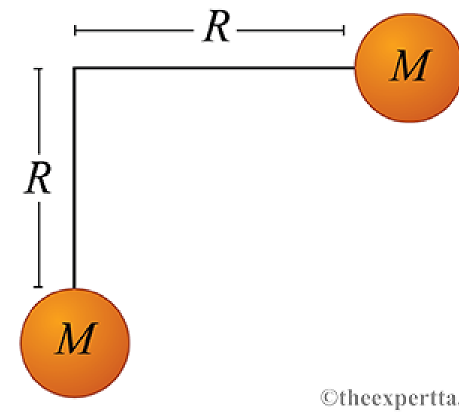
We can apply the formula for tangential velocity, substituting in the expression above:

$$\begin{aligned} v_t &= r\omega \\ &= L\sqrt{\frac{3g\sin\phi}{L}} \\ &= \sqrt{3gL\sin\phi} \\ &= \sqrt{3(9.81\ \text{m/s}^2)(1.5\ \text{m})\sin(20.1^\circ)} \\ &= 3.895\ \text{m/s} \end{aligned}$$

$$\boxed{v_t = 3.895\ \text{m/s}}$$

Problem 15:

Consider two equal masses M attached to the ends of massless rods, of length R, as shown in the figure. Treat the two masses as point masses, each located at a distance R from the point where the rods touch.



Part (a) **What is the moment of inertia of this system, about an axis perpendicular to the page and passing through the point where the rods touch?**

The total moment of inertia of a system is the sum of the individual moments of inertia. The moment of inertia for a mass, M in kg, a distance R, in m, from a pivot point is

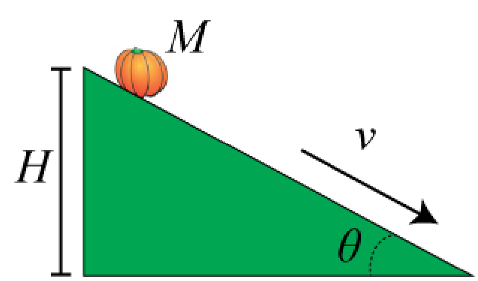
$$I_{PointMass} = MR^2 \text{ kg m}^2$$

Since there are two equal masses a distance R from the pivot point, the total moment of inertia is the sum of these two moments of inertia.

$$\boxed{2MR^2 \text{ kg m}^2}$$

Problem 16:

You have a pumpkin of mass M and radius R. The pumpkin has the shape of a sphere, but it is not uniform inside so you do not know its moment of inertia.
In order to determine the moment of inertia, you decide to roll the pumpkin down an incline that makes an angle θ with the horizontal. The pumpkin starts from rest and rolls without slipping. When it has descended a vertical height H, it has acquired a speed of $v = \sqrt{(5gH/4)}$.



Part (a) **Find the moment of inertia I of the pumpkin in terms of M and R.**

The initial total energy of the pumpkin equal to its gravitational potential energy. This energy is equal to the total energy the pumpkin has at the bottom of the hill, which is translational and rotational kinetic energy,

energy at bottom of hill = energy at the initial height

$$\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = MgH$$

where $\omega = \frac{v}{R}$. We are assuming here that the pumpkin is approximately spherical and it rolls without slipping.

$$Mv^2 + I\omega^2 = 2MgH$$

$$Mv^2 + I\left(\frac{v}{R}\right)^2 = 2MgH$$

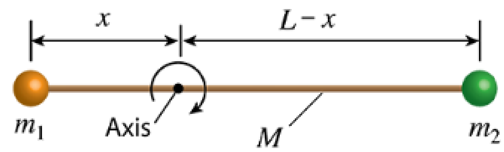
Solving for the moment of inertia gives,

$$\begin{aligned} I &= \frac{(2MgH - Mv^2)}{\left(\frac{v}{R}\right)^2} \\ &= \frac{\left(2MgH - M\left(\sqrt{\frac{5gh}{4}}\right)^2\right)}{\left(\frac{\left(\sqrt{\frac{5gh}{4}}\right)}{R}\right)^2} \\ &= \frac{\left(2MgH - \frac{5MgH}{4}\right)}{\left(\frac{5gH}{4R^2}\right)} \\ &= \frac{(8MgHR^2 - 5MgHR^2)}{(5gh)} \end{aligned}$$

$$\boxed{I = \frac{3}{5}MR^2}$$

Problem 17:

Two small spheres, with masses m_1 and m_2 are attached to the ends of a rod of mass M and length L. The system rotates about an axis that is a distance x from the sphere with mass m_1 as shown in the diagram. The system is initially at rest, but then a constant torque is applied until the angular speed of the system attains a specified value.



Part (a) **Write an expression for x so that the work done by the applied torque to achieve the specified angular speed has its smallest possible value. Treat the spheres as point masses. Your answer will be in terms of m_1, m_2, M, and L.**

According to the work-energy theorem, the net work done is equal to the change in kinetic energy. Since the system starts from rest, the work by a torque to accelerate it to an angular speed of ω is given by

$$W = \Delta K_R = \frac{1}{2}I\omega^2 - 0 = \frac{1}{2}I\omega^2 \ .$$

From the equation above, we see that the work will have its smallest possible value when the moment of inertia, I, has its smallest possible value. To determine the value of x that gives the smallest possible value of I we can set the derivative of I with respect to x equal to 0 and solve for x.

First we must determine the total moment of inertia of the system. The moments of inertia from the two point masses are as follows:

$$\begin{aligned} I_{m_1} &= m_1 x^2 \\ I_{m_2} &= m_2(L - x)^2 \end{aligned}$$

To determine the moment of inertia of the rod, we can use the parallel axis theorem:

$$I_{rod} = I_{CM} + Md^2$$

In this case, the moment of inertia of of the rod with the axis through its center of mass is

$$I_{CM} = \frac{1}{12}ML^2 \ ,$$

and the distance from the axis of rotation from the center of mass is

$$d = \frac{L}{2} - x \ .$$

The total moment of inertial of the system is then

$$
\begin{aligned}
I &= I_{m_1} + I_{m_2} + I_{rod} \\
&= m_1 x^2 + m_2(L-x)^2 + \frac{1}{12}ML^2 + M\left(\frac{L}{2} - x\right)^2 \\
&= m_1 x^2 + m_2(L^2 - 2Lx + x^2) + \frac{1}{12}ML^2 + M\left(\frac{L^2}{4} - Lx + x^2\right) \\
&= m_1 x^2 + m_2 L^2 - 2m_2 Lx + m_2 x^2 + \frac{1}{3}ML^2 - MLx + Mx^2
\end{aligned}
$$

Setting the derivative of I with respect to x equal to 0 and solving for x:

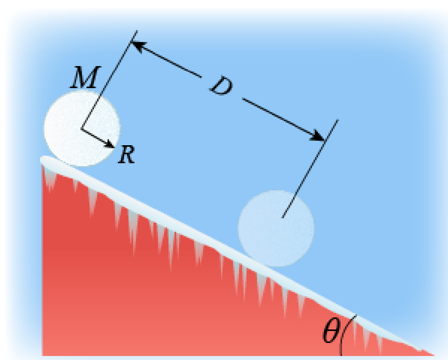
$$
\begin{aligned}
0 &= \frac{dI}{dx} \\
&= \frac{d}{dx}\left[m_1 x^2 + m_2 L^2 - 2m_2 Lx + m_2 x^2 + \frac{1}{3}ML^2 - MLx + Mx^2\right] \\
&= 2m_1 x - 2m_2 L + 2M_2 x - ML + 2Mx
\end{aligned}
$$

$$
\begin{aligned}
2m_1 x + 2m_2 x + 2Mx &= 2m_2 L + ML \\
(m_1 + m_2 + M)x &= m_2 L + \frac{1}{2}ML \\
x &= \frac{(m_2 + \frac{1}{2}M)L}{m_1 + m_2 + M}
\end{aligned}
$$

$$
\boxed{x = \left(\frac{m_2 + \frac{1}{2}M}{m_1 + m_2 + M}\right)L}
$$

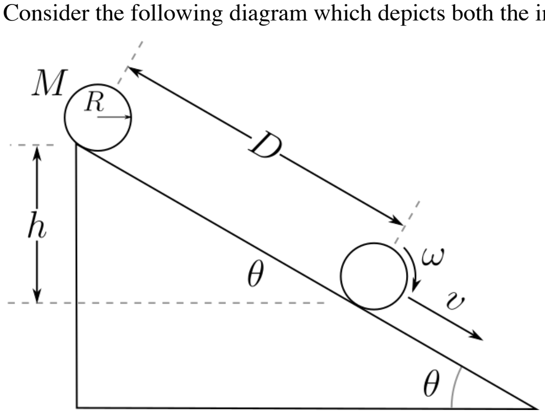
Problem 18:

A spherical snowball of mass M and radius R starts from rest and rolls without slipping down a roof that makes an angle θ with the horizontal.



Part (a) **What is the angular speed, ω, of the snowball after it has traveled a distance D down the sloped roof?**

Consider the following diagram which depicts both the initial and the final position of the snowball on the sloped roof.



As the snowball travels a distance D parallel to the incline of the roof, its elevation decreases by a distance h which may be obtained as

$$h = D \sin\theta$$

We are free to choose the height at which the gravitational potential energy is zero, so let's choose it to be the final location of the snowball. Then

$$U_{initial} = Mgh$$
$$U_{final} = 0$$

The condition of rolling without slipping is very important. As the snowball moves down the roof, friction does not dissipate any of the mechanical energy. The linear acceleration is reduced, but a torque due to friction causes an angular acceleration. The condition for rolling without slipping relates the angular and linear kinematic variables as

$$a = R\alpha$$
$$v = R\omega$$

The initial potential energy will be converted to a mixture of linear and rotational kinetic energy.

$$U_{initial} = K_{linear} + K_{rot.}$$
$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

We note that the moment of inertia for solid sphere is given by

$$I = \frac{2}{5}MR^2$$

We now make substitutions for the height, the linear speed, and the moment of inertia.

$$MgD\sin\theta = \frac{1}{2}M(R\omega)^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\omega^2$$

The mass now cancels, and

$$gD\sin\theta = \frac{7}{10}R^2\omega^2$$

or

$$\omega^2 = \frac{10gD\sin\theta}{7R^2}$$

$$\boxed{\omega = \sqrt{\frac{10gD\sin\theta}{7R^2}}}$$