Hence,

$$\int_1^t \frac{d}{ds} e^{s^2} y(s)\, ds = \int_1^t s e^{s^2}\, ds$$

so that

$$e^{s^2} y(s)\Big|_1^t = \frac{e^{s^2}}{2}\Big|_1^t.$$

Consequently,

$$e^{t^2} y - 2e = \frac{e^{t^2}}{2} - \frac{e}{2}$$

and

$$y = \frac{1}{2} + \frac{3e}{2} e^{-t^2} = \tfrac{1}{2} + \tfrac{3}{2} e^{1-t^2}.$$

Example 7. Find the solution of the initial-value problem

$$\frac{dy}{dt} + y = \frac{1}{1+t^2}, \quad y(2) = 3.$$

Solution. Here $a(t) = 1$, so that

$$\mu(t) = \exp\left(\int a(t)\, dt\right) = \exp\left(\int 1\, dt\right) = e^t.$$

Multiplying both sides of the equation by $\mu(t)$ we obtain that

$$e^t\left(\frac{dy}{dt} + y\right) = \frac{e^t}{1+t^2} \quad \text{or} \quad \frac{d}{dt} e^t y = \frac{e^t}{1+t^2}.$$

Hence

$$\int_2^t \frac{d}{ds} e^s y(s)\, ds = \int_2^t \frac{e^s}{1+s^2}\, ds,$$

so that

$$e^t y - 3e^2 = \int_2^t \frac{e^s}{1+s^2}\, ds$$

and

$$y = e^{-t}\left[3e^2 + \int_2^t \frac{e^s}{1+s^2}\, ds\right].$$

Exercises

In each of Problems 1–7 find the general solution of the given differential equation.

1. $\dfrac{dy}{dt} + y\cos t = 0$

2. $\dfrac{dy}{dt} + y\sqrt{t}\, \sin t = 0$

3. $\frac{dy}{dt}+\frac{2t}{1+t^2}y=\frac{1}{1+t^2}$

4. $\frac{dy}{dt}+y=te^t$

5. $\frac{dy}{dt}+t^2y=1$

6. $\frac{dy}{dt}+t^2y=t^2$

7. $\frac{dy}{dt}+\frac{t}{1+t^2}y=1-\frac{t^3}{1+t^4}y$

In each of Problems 8–14, find the solution of the given initial-value problem.

8. $\frac{dy}{dt}+\sqrt{1+t^2}\,y=0, \quad y(0)=\sqrt{5}$

9. $\frac{dy}{dt}+\sqrt{1+t^2}\,e^{-t}y=0, \quad y(0)=1$

10. $\frac{dy}{dt}+\sqrt{1+t^2}\,e^{-t}y=0, \quad y(0)=0$

11. $\frac{dy}{dt}-2ty=t, \quad y(0)=1$

12. $\frac{dy}{dt}+ty=1+t, \quad y(\tfrac{3}{2})=0$

13. $\frac{dy}{dt}+y=\frac{1}{1+t^2}, \quad y(1)=2$

14. $\frac{dy}{dt}-2ty=1, \quad y(0)=1$

15. Find the general solution of the equation

$$(1+t^2)\frac{dy}{dt}+ty=(1+t^2)^{5/2}.$$

(*Hint*: Divide both sides of the equation by $1+t^2$.)

16. Find the solution of the initial-value problem

$$(1+t^2)\frac{dy}{dt}+4ty=t, \qquad y(1)=\tfrac{1}{4}.$$

17. Find a continuous solution of the initial-value problem

$$y'+y=g(t), \qquad y(0)=0$$

where

$$g(t)=\begin{cases}2, & 0\leqslant t\leqslant 1\\ 0, & t>1\end{cases}.$$

18. Show that every solution of the equation $(dy/dt)+ay=be^{-ct}$ where a and c are positive constants and b is any real number approaches zero as t approaches infinity.

19. Given the differential equation $(dy/dt)+a(t)y=f(t)$ with $a(t)$ and $f(t)$ continuous for $-\infty<t<\infty$, $a(t)\geqslant c>0$, and $\lim_{t\to\infty}f(t)=0$, show that every solution tends to zero as t approaches infinity.

When we derived the solution of the nonhomogeneous equation we tacitly assumed that the functions $a(t)$ and $b(t)$ were continuous so that we could perform the necessary integrations. If either of these functions was discontinuous at a point t_1, then we would expect that our solutions might be discontinuous at $t=t_1$. Problems 20–23 illustrate the variety of things that