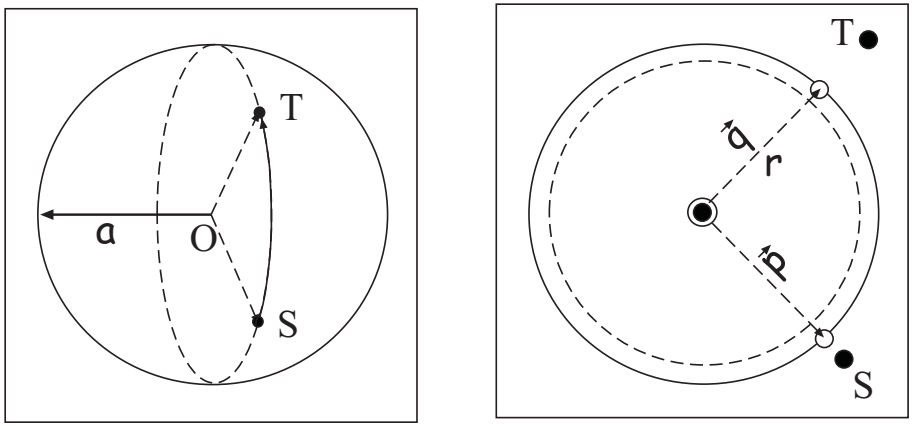
3D Rotation User Interfaces and Specifying 3D Orientations

Goal: Use of a 2-button or 3-button MOUSE to orient a model/camera in 3D world coordinates

Solutions:

- Virtual Sphere or Arcball
- Rotations easier to specify as a vector (axis) in 3D and angle
- Composing Rotations about any vector in 3D,
- Quaternions as alternative to Euler Angles

The Arcball



1. Choose region of screen as projection of sphere: 2D point O center, radius ρ.
2. Get initial 2D point S on button-down.
3. Compute $\vec{s} = (s_x, s_y)$.
4. Compute 2D radius: $r^2 = s_x^2 + s_y^2$.
5. Map 2D vector $\vec{s}$ to 3D unit vector $\vec{p}$ as before:

If $r^2 > \rho^2$, map to silhouette of unit sphere:

$$p_x \leftarrow s_x/r$$
$$p_y \leftarrow s_y/r$$
$$p_z \leftarrow 0.$$

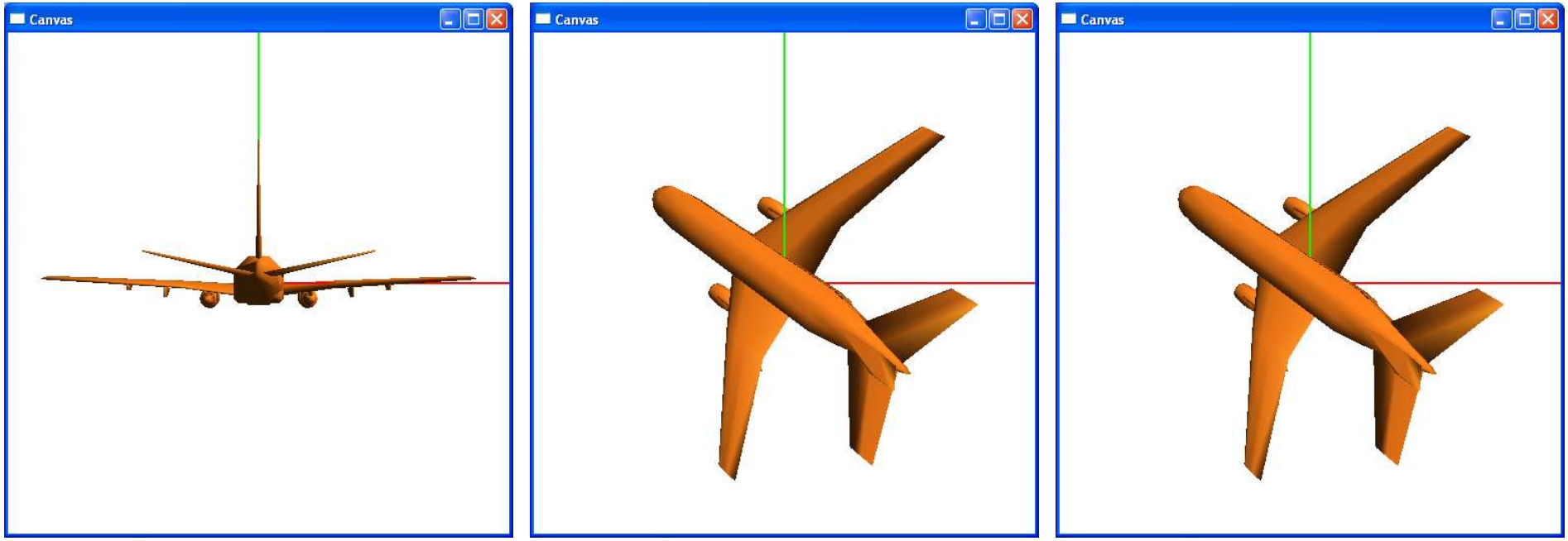
Else,

$$p_x \leftarrow s_x/\rho$$
$$p_y \leftarrow s_y/\rho$$
$$p_z \leftarrow \sqrt{1 - p_x^2 - p_y^2}.$$

6. For each new mouse position T, map to unit 3D vector $\vec{q}$ as above.
7. Axis: $\vec{a} = \vec{p} \times \vec{q}$.
8. Angle: $\theta = 2\cos^{-1}(\vec{p} \cdot \vec{q})$.
9. Notes:
   - Rotation given by *twice* the angle of the great arc between $\vec{p}$ and $\vec{q}$.

- Doubling the angle matches orientation's mathematical structure better.
- Points on opposite sides of the sphere silhouette allow a rotation by 360°.

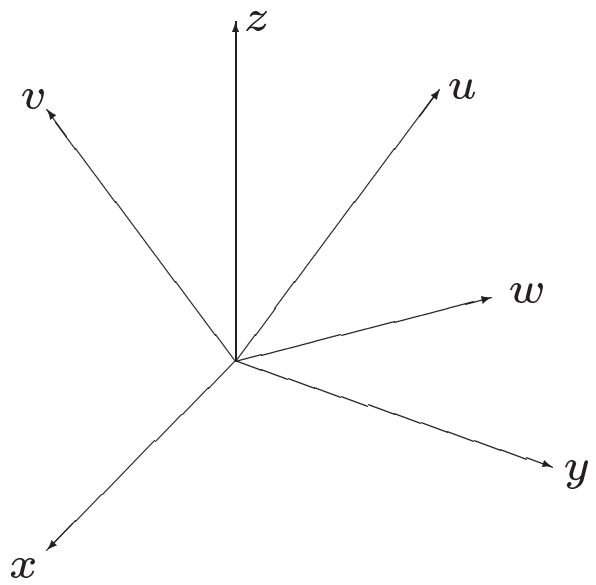
Specifying Orientation using Quaternions are easier than Euler angles

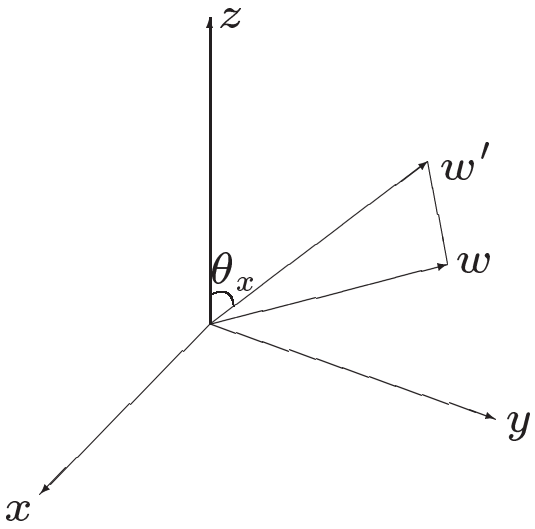



```
glMatrixMode (GL_MODELVIEW);
glRotatef(45, 1, 0, 0);
glRotatef(45, 0, 1, 0);
glRotatef(45, 0, 0, 1);
glutSolidTeapot(1);
```

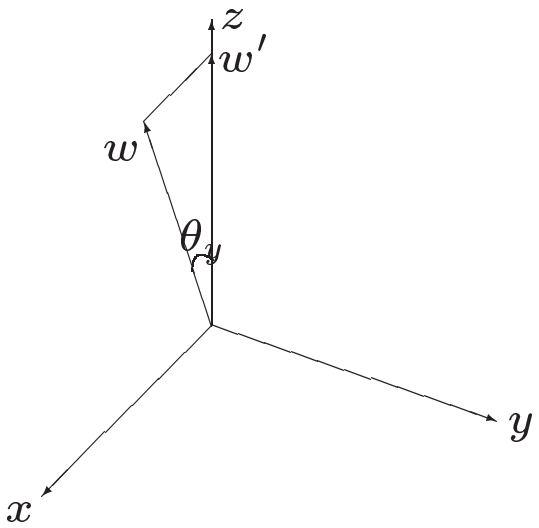
```
Quaternion = -0.46, -0.21, -0.41, 0.75
Rotation Matrix =
   0.55  0.82  0.06  0.00
  -0.43  0.22  0.87  0.00
   0.71 -0.51  0.48  0.00
   0.00  0.00  0.00  1.00
```

Orientations and Quaternions

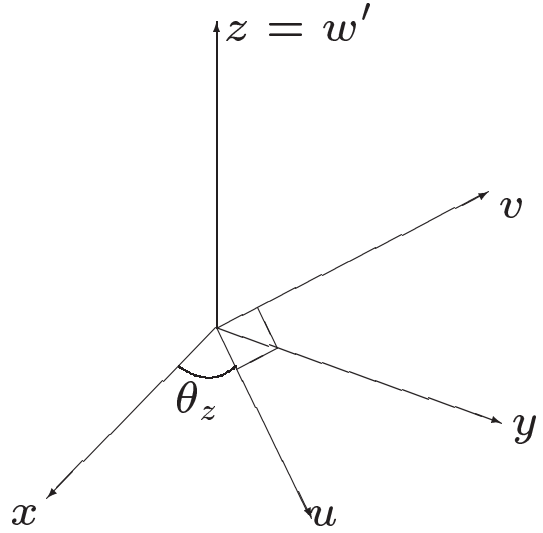




(a) Rotate about x so that w lies in xz-plane.

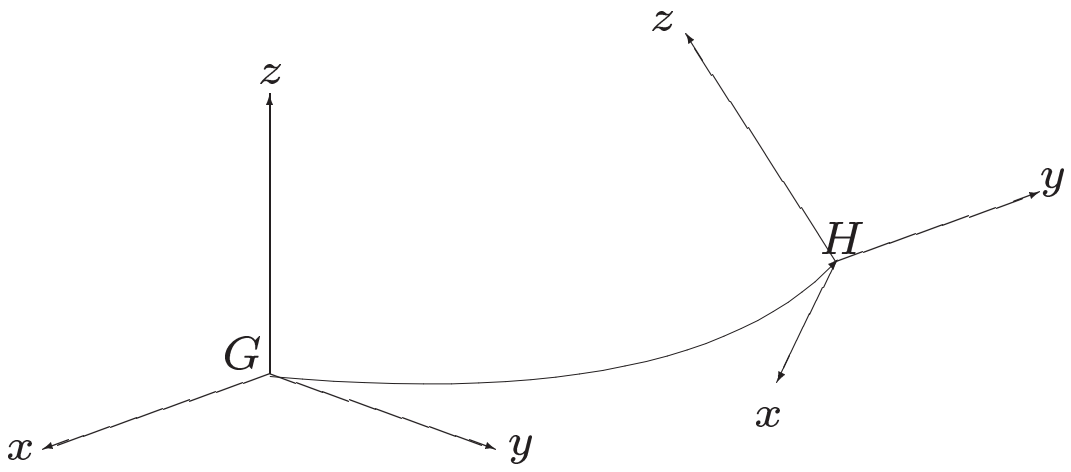


(b) Rotate about y so that w coincides with z.



(c) Rotate about z so that u and v coincide with x and y.

Rotating a frame to coincide with the standard frame



Smooth Interpolation of Frames

It is possible to perform any change of orientation about an arbitrary axis with three rotations, one about each of the coordinate axes, by a triple of three angles, $(\theta_x, \theta_y, \theta_z)$. These define a general rotation matrix, by composing the three basic rotations:

$$\mathbf{R}(\theta_x, \theta_y, \theta_z) = \mathbf{R}_z(\theta_z)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x).$$

These three angles are called the **Euler angles** for the rotation. Thus, we can parameterize

any rotation in 3-space as triple of numbers, each in the range $\alpha \in [0, 2\pi]$.

With $c_a = \cos(\theta_a)$ and $s_a = \sin(\theta_a)$,

$$R(\theta_x, \theta_y, \theta_z) = \begin{pmatrix} c_y c_z & c_y s_z & -s_y & 0 \\ s_x s_y c_z - c_x s_z & s_x s_y s_z + c_x c_z & s_x c_y & 0 \\ c_x s_y c_z + s_x s_z & c_x s_y s_z - s_x c_z & c_x c_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= R_z(\theta_z) R_y(\theta_y) R_x(\theta_x),$$

where $R_x(\theta_x)$, $R_y(\theta_y)$ and $R_z(\theta_z)$ are the standard rotation matrices.

Given a point P represented as a homogeneous row vector, the rotation of P is given by $P' = PR(\theta_x, \theta_y, \theta_z)$. Animation between two rotations involves interpolating independently the three angles θ_x, θ_y and θ_z.

The standard rotation matrices are given by

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_x & s_x & 0 \\ 0 & -s_x & c_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\theta_x) = \begin{pmatrix} c_y & 0 & -s_y & 0 \\ 0 & 1 & 0 & 0 \\ s_y & 0 & c_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\theta_x) = \begin{pmatrix} c_z & s_z & 0 & 0 \\ -s_z & c_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Gimbal Lock

An example will clarify the **parametric singularity** problem, commonly known as *gimbal lock*. Gimbal lock is a mechanical problem that arises in the support of gyroscopes by three

nested rotating frames.

Suppose we set $\theta_y = \pi/2 = 90^\circ$, and set θ_x and θ_z arbitrarily. Then $c_y = 0$, $s_y = 1$ and the matrix $R(\theta_x, \pi/2, \theta_z)$ can be reduced to

$$R(\theta_x, \theta_y, \theta_z) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ s_x c_z - c_x s_z & s_x s_z + c_x c_z & 0 & 0 \\ c_x c_z + s_x s_z & c_x s_z - s_x c_z & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 & 0 \\ \sin(\theta_x - \theta_z) & \cos(\theta_x - \theta_z) & 0 & 0 \\ \cos(\theta_x - \theta_z) & \sin(\theta_x - \theta_y) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Transformation only depends on the difference $\theta_x - \theta_z$, and hence only has one degree of freedom when it should have two.

This occurs because a y-roll by $\pi/2$ rotates the x-axis onto the negative z axis, and so a x-roll by θ has the same effect as a z-roll by $-\theta$. Gimbal lock can be very frustrating in practice:

- During interactive manipulation the object will seem to "stick";
- Certain orientations can be hard to obtain if approached from the wrong direction;
- Interpolation through these parametric singularities will behave strangely.

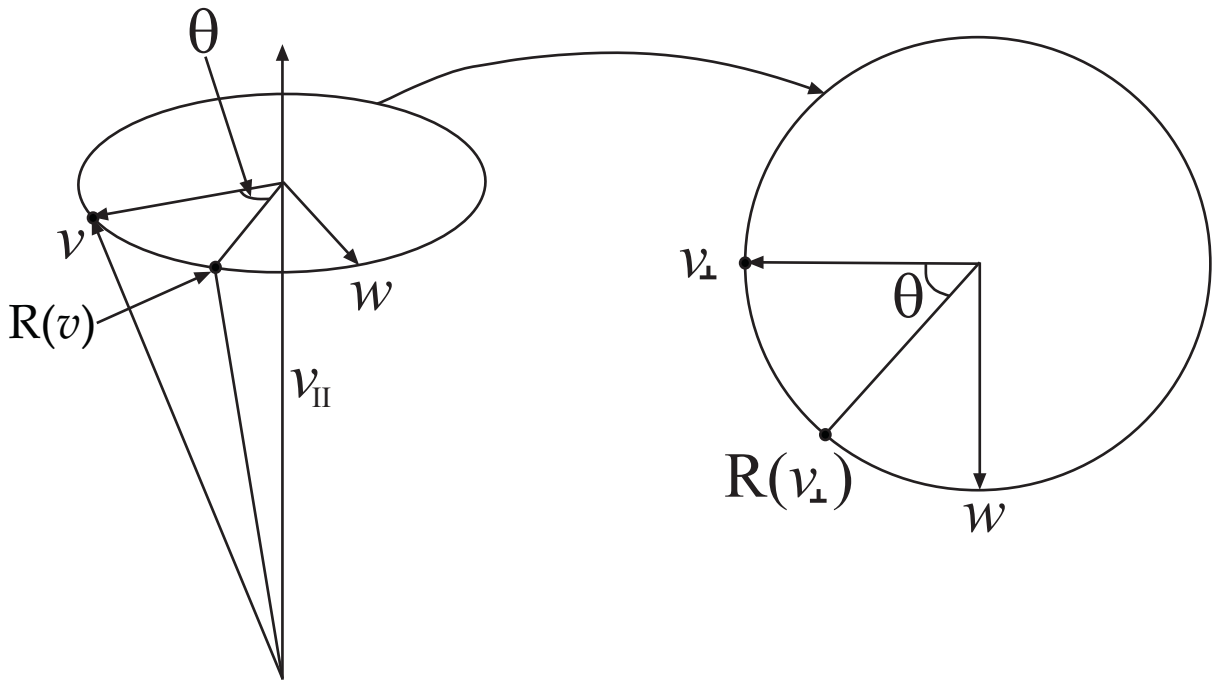
Perhaps a somewhat more natural way to express rotations (about the origin) in 3-space is in term of two quantities, $(\theta, \vec{u})$, consisting of an angle θ, and an axis of rotation $\vec{u}$. Let's consider how we might do this. First consider a vector $\vec{v}$ to be rotated to $R(\vec{v})$. Let us assume that $\vec{u}$ is of unit length.

In order to derive this, we begin by decomposing $\vec{v}$ as the sum of its components that are parallel to and orthogonal to $\vec{u}$, respectively.

$$\vec{v}_{\parallel} = (\vec{u} \cdot \vec{v})\vec{u}$$
$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel} = \vec{v} - (\vec{u} \cdot \vec{v})\vec{u}.$$

Note that $\vec{v}_{\parallel}$ is unaffected by the rotation, but $\vec{v}_{\perp}$ is rotated to a new position $R(\vec{v}_{\perp})$. To determine this rotated position, we will first construct a vector that is orthogonal to $\vec{v}_{\perp}$ lying in the plane of rotation,

$$\vec{w} = \vec{u} \times \vec{v}_{\perp} = \vec{u} \times (\vec{v} - \vec{v}_{\parallel}) = (\vec{u} \times \vec{v}) - (\vec{u} \times \vec{v}_{\parallel}) = \vec{u} \times \vec{v}.$$



Angular displacement

Now, consider the plane spanned by $\vec{v}_\perp$ and $\vec{w}$. We have

$$R(\vec{v}_\perp) = (\cos\theta)\vec{v}_\perp + (\sin\theta)\vec{w}.$$

From this we have

$$\begin{aligned} R(\vec{v}) &= R(\vec{v}_{\parallel}) + R(\vec{v}_{\perp}) \\ &= R(\vec{v}_{\parallel}) + (\cos\theta)\vec{v}_{\perp} + (\sin\theta)\vec{w} \\ &= (\vec{u}\cdot\vec{v})\vec{u} + (\cos\theta)(\vec{v} - (\vec{u}\cdot\vec{v})\vec{u}) + (\sin\theta)\vec{w} \\ &= (\cos\theta)\vec{v} + (1-\cos\theta)\vec{u}(\vec{u}\cdot\vec{v}) + (\sin\theta)(\vec{u}\times\vec{v}). \end{aligned}$$

Quaternions:

$$i^2 = j^2 = k^2 = -1 \quad ij = k, \quad jk = i, \quad ki = j.$$

Combining these, it follows that $ji = -k$, $kj = -i$ and $ik = -j$. A quaternion is defined to be a generalized complex number of the form

$$q = q_0 + q_1 i + q_2 j + q_3 k.$$

We will see that quaternions bear a striking resemblance to our notation for angular displacement. In particular, we can rewrite the quaternion notation in terms of a scalar and vector as

$$q = (s, \vec{u}) = s + u_x i + u_y j + u_z k.$$

Furthermore define the product of quaternions to be

$$q_1 q_2 = (s_1 s_2 - (\vec{u}_1 \cdot \vec{u}_2), \quad s_1 \vec{u}_2 + s_2 \vec{u}_1 + \vec{u}_1 \times \vec{u}_2).$$

Define the conjugate of a quaternion $q = (s, \vec{u})$ to be $\bar{q} = (s, -\vec{u})$. Define the magnitude of a quaternion to be the square root of this product:

$$|q|^2 = q\bar{q} = s^2 + |\vec{u}|^2.$$

A unit quaternion is one of unit magnitude, $|q| = 1$. A pure quaternion is one with a 0 scalar component

$$p = (0, \vec{v}).$$

Any quaternion of nonzero magnitude has a multiplicative inverse, which is

$$q^{-1} = \frac{1}{|q|^2}\bar{q}.$$

Quaternion and Rotation:

Define the rotation operator

$$R_q(p) = qpq^{-1}.$$

$$R_q(p) = (0, (s^2 - (\vec{u} \cdot \vec{u}))\vec{v} + 2\vec{u}(\vec{u} \cdot \vec{v}) + 2s(\vec{u} \times \vec{v})).$$

Unit quaternions can be shown to be isomorphic to orientations and given by

$$q = (\cos\theta, (\sin\theta)\vec{u}), \qquad \text{where } |\vec{u}| = 1.$$

This is equivalent to a rotation by an angle 2θ around the axis $\vec{u}$.

Plugging q into the above expression $R_q(p)$, we have

$$\begin{aligned} R_q(p) &= (0, (\cos^2\theta - \sin^2\theta)\vec{v} + 2(\sin^2\theta)\vec{u}(\vec{u} \cdot \vec{v}) + 2\cos\theta\sin\theta(\vec{u} \times \vec{v})) \\ &= (0, (\cos 2\theta)\vec{v} + (1 - \cos 2\theta)\vec{u}(\vec{u} \cdot \vec{v}) + \sin 2\theta(\vec{u} \times \vec{v})). \end{aligned}$$

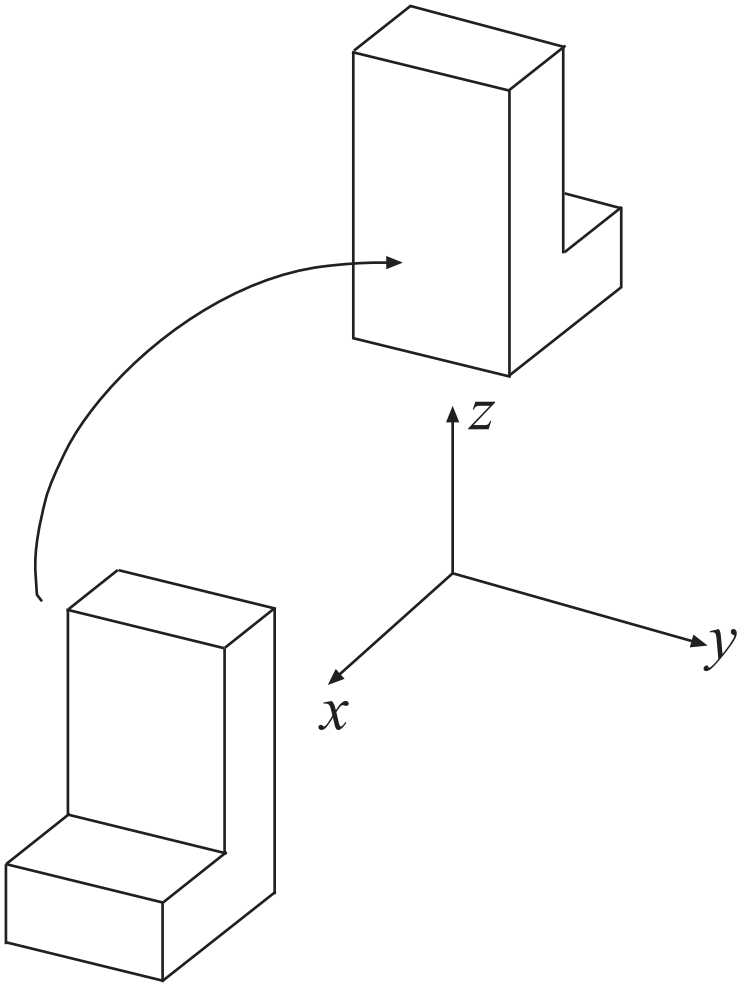
Thus, in summary, we encode points in 3-space as pure quaternions

$$p = (0, \vec{v}),$$

and we encode a rotation by angle q about a unit vector $u \rightarrow$ as a unit quaternion

$$q = (\cos(\theta/2), \sin(\theta/2)\vec{u}),$$

then the image of the point under this rotation is given by the vector part of the result of the quaternion rotation operator $R_q(p)$.



Rotation example.

Composing Rotations:

Given two unit quaternions q and q', a rotation by q followed by a rotation by q' is equivalent to a single rotation by the product $q'' = q'q$. That is,

$$R_{q'}R_q = R_{q''} \qquad \text{where } q'' = q'q.$$

This follows from the associativity of quaternion multiplication, and the fact that $(qq')^{-1} = q^{-1}q'^{-1}$, as shown below.

$$\begin{aligned} R_{q'}(R_q(p)) &= q'(qpq - 1)q'^{-1} \\ &= (q'q)p(q^{-1}q'^{-1}) \\ &= (q'q)p(qq')^{-1} \\ &= q''pq''^{-1} \\ &= R_{q''}(p). \end{aligned}$$

Matrices and Quaternions:

Given a unit quaternion

$$q = (\cos(\theta/2), \sin(\theta/2)\vec{u}) = (w, (x, y, z))$$

what is the corresponding affine transformation (expressed as a rotation matrix). By simply expanding the definition of $R_q(p)$, it is not hard to show that the following (homogeneous) matrix is equivalent

$$\begin{pmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy & 0 \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx & 0 \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

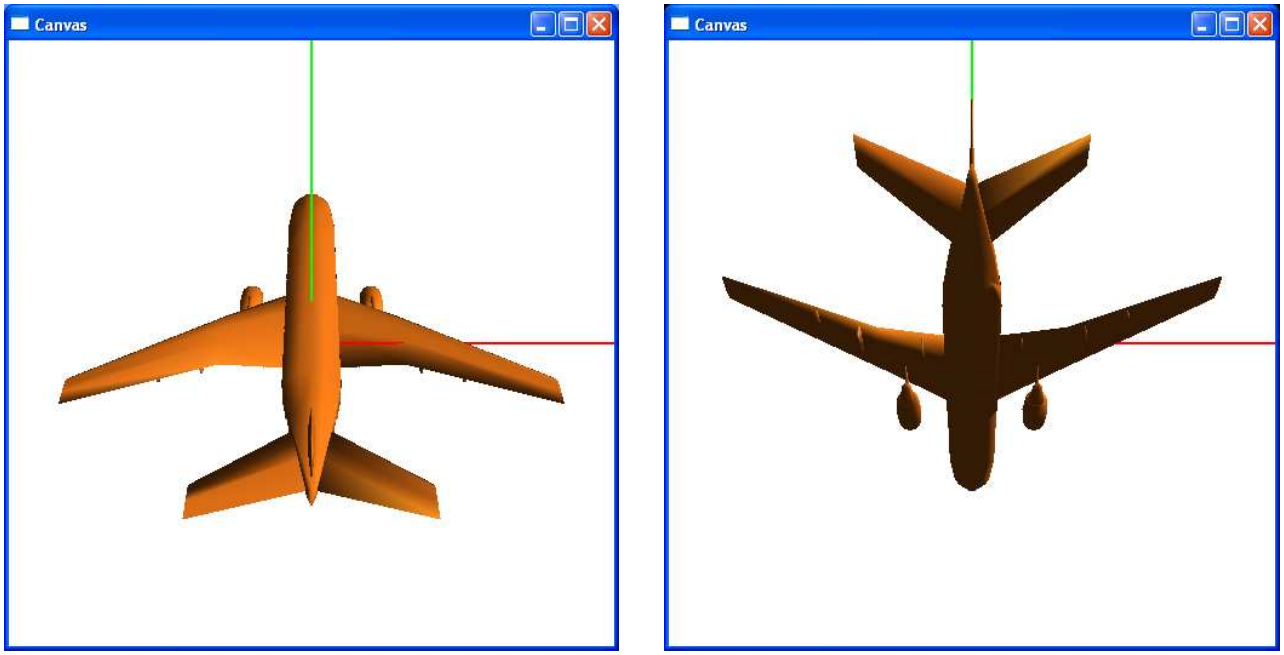
To convert from an orthogonal rotation matrix to a unit quaternion, we observe that if $M = [m_{i,j}]$ is the affine transformation in homogeneous form,

$$\text{trace}(M) = 4 - 4(x^2 + y^2 + z^2) = 4w^2.$$

Once we have w, we can find the order quantities by cancelling symmetric terms:

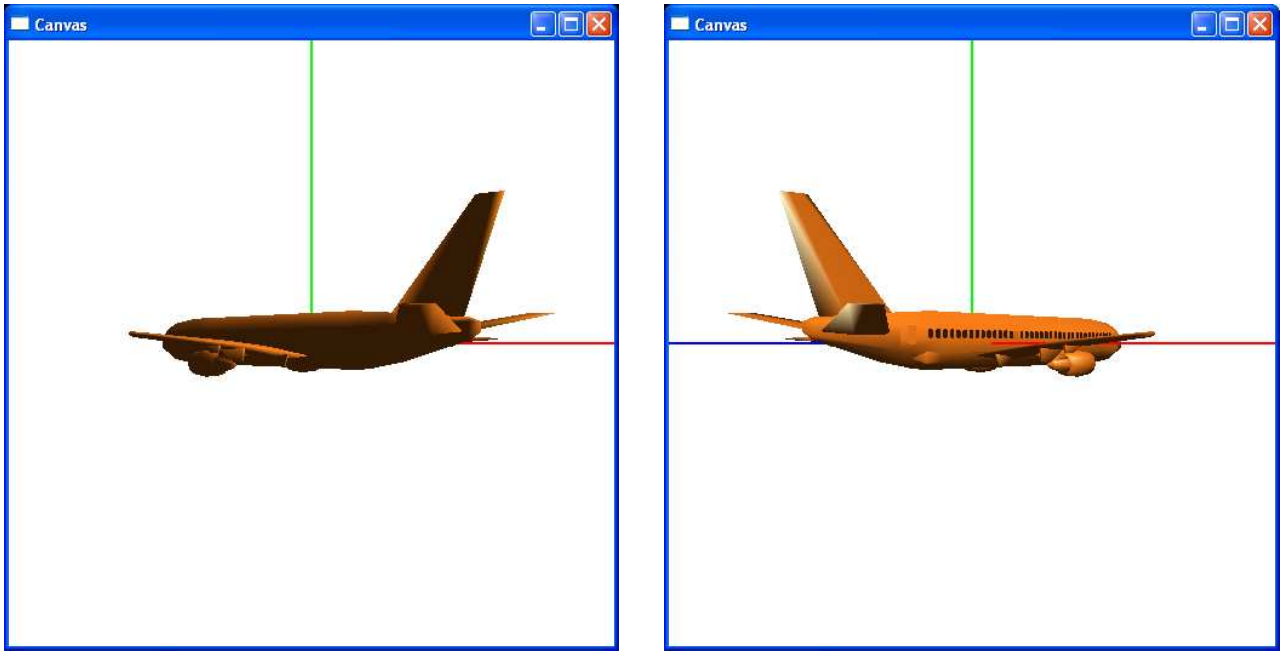
$$x = \frac{m_{32} - m_{23}}{4w},$$
$$y = \frac{m_{13} - m_{31}}{4w},$$
$$z = \frac{m_{21} - m_{12}}{4w}$$

Pitch



Rotation Axis = 1, 0, 0, 0
Rotation Angle = $\pm\pi/4$
Quaternion Vector = $\pm$0.382683 0.000000 0.000000 0.923880

Yaw

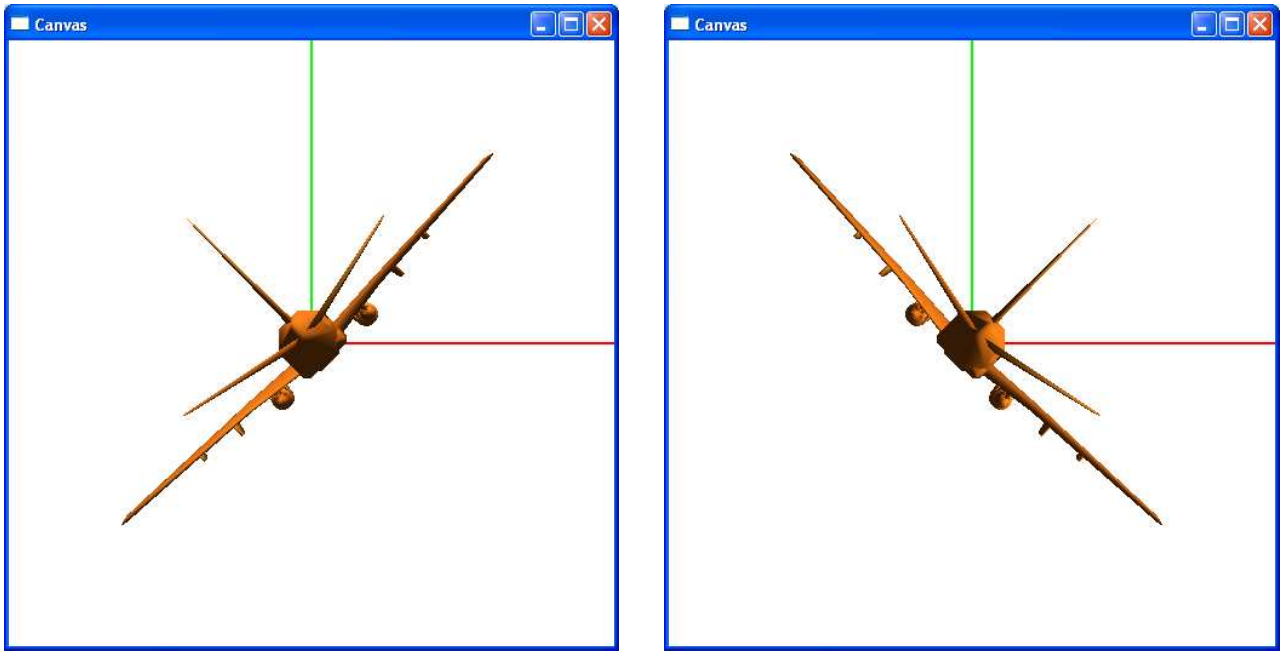


Rotation Axis = 0, 1, 0, 0

Rotation Angle = $\pm\pi/4$

Quaternion Vector = 0.000000 $\pm$0.382683 0.000000 0.923880

Roll



Rotation Axis = 0, 0, 1, 0
Rotation Angle = $\pm\pi/4$
Quaternion Vector = 0.000000 0.000000 $\pm$0.382683 0.923880

Additonal Examples

Matrix and Quaternion FAQ
http://skal.planet-d.net/demo/matrixfaq.htm

The following web-page contains C++ source codes:
http://www.lboro.ac.uk/departments/ma/gallery/quat/intro.html

Reading Assignment and News

Chapter 4 pages 212 - 228, of Recommended Text.

Please also track the News section of the Course Web Pages for the most recent Announcements related to this course.

(http://www.cs.utexas.edu/users/bajaj/graphics25/cs354/)