Fractals

Consider a complex number $z = a + bi$ as a point (a, b) or vector in the Real Euclidean plane [1, i] with modulus $|z|$ the length of the vector and equal to $\sqrt{a^2 + b^2}$.

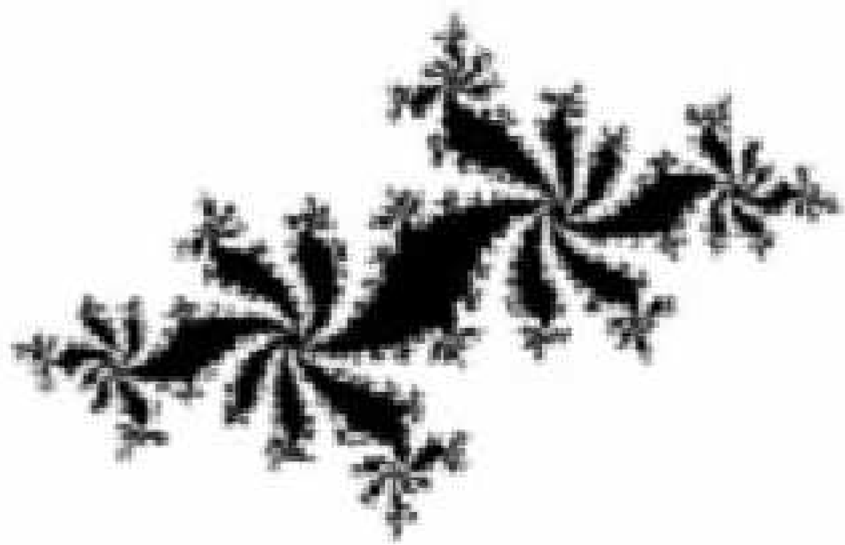
Complex arithmetic rules:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

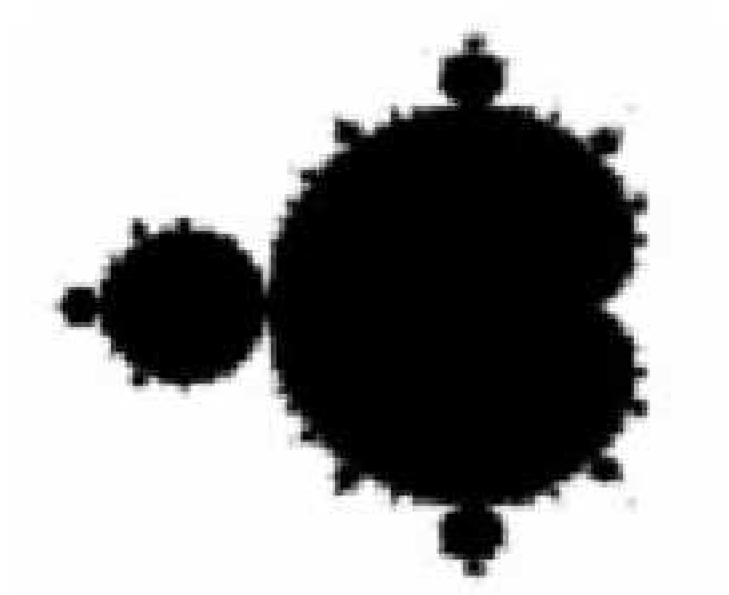
$z \longrightarrow z^2$

All numbers with modulus 1 will stay at modulus 1 and is the *attractor set* or *fixed-point* of this **iterated function system**.

Julia Set for the point c: The attractor set of the iterated function system $z \longrightarrow z^2 + c$ with c a complex constant



Julia Set for $c = -0.62 - 0.44i$



Mandelbrot Set: Color the point c black if **Julia** (c) is connected, and $white$ otherwise.

Fractal Dimension:

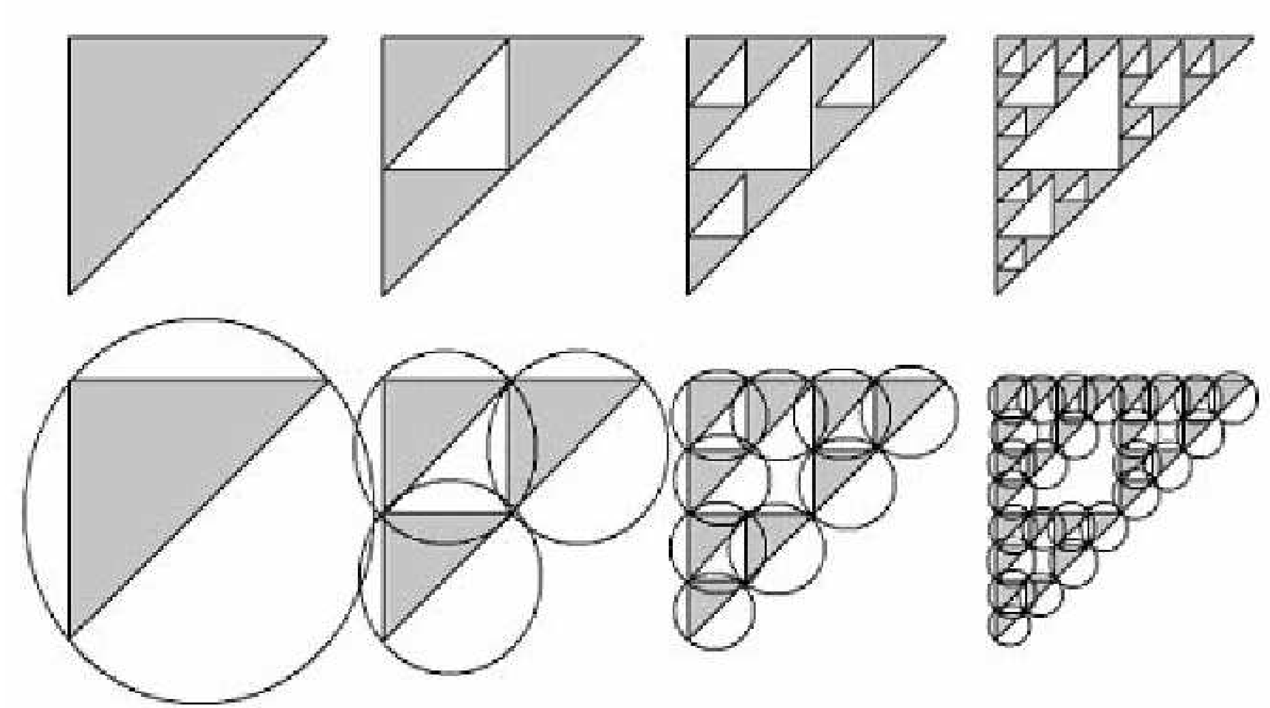
$$N(A, \epsilon) = \text{smallest number of } \epsilon\text{-balls needed to cover } A.$$

Object A has dimension d if $N(A, \epsilon)$ grows as $C(1/\epsilon)^d$ for constant C

$$\text{Fractal dimension } d = \lim_{\epsilon \to 0} \frac{\ln N(A, \epsilon)}{\ln(1/\epsilon)}$$

A **fractal** is an object which is *self-similar at different scales* and has a *non-integer fractal dimension*

$$\begin{aligned} d &= \lim_{\epsilon \to 0} \frac{\ln N(A, \epsilon)}{\ln(1/\epsilon)} \\ &= \lim_{k \to \infty} \frac{\ln N(A, (1/2^k))}{\ln(1/(1/2^k))} \\ &= \lim_{k \to \infty} \frac{\ln 3^k}{\ln 2^k} = \lim_{k \to \infty} \frac{k \ln 3}{k \ln 2} \\ &= \lim_{k \to \infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2} \approx 1.58496. \end{aligned}$$



The Sierpinski triangle covered by 3^k $(1/2^k)$-balls

Repeated Subdivision rule:

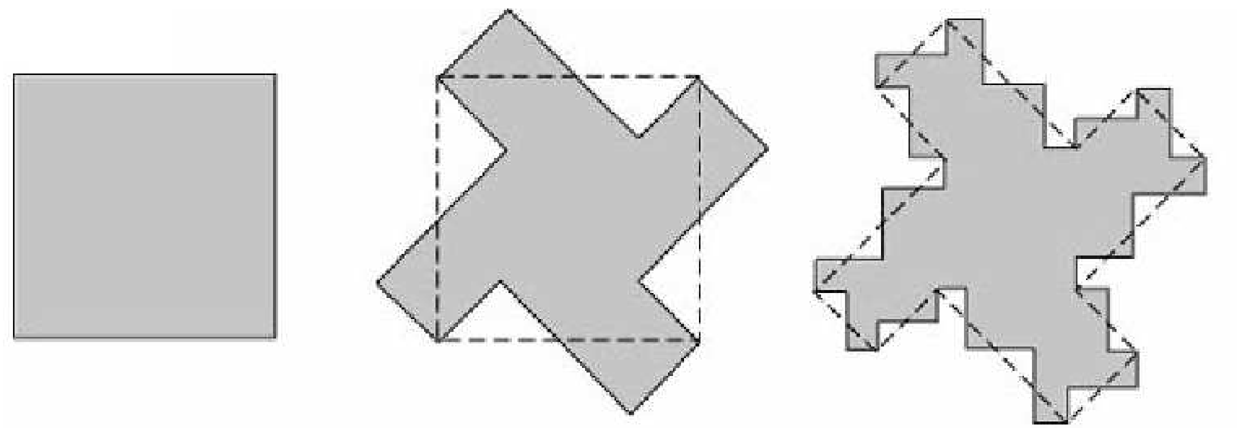
Replace each piece of length x by b nonoverlapping piece of length x/a.

Fractal dimension is

$$d = \frac{\ln b}{\ln a}$$

For object below the area doesn't change but boundary length does. The fractal dimension is

$$\frac{\ln 4}{\ln(2\sqrt{2})} = 1.3333.$$

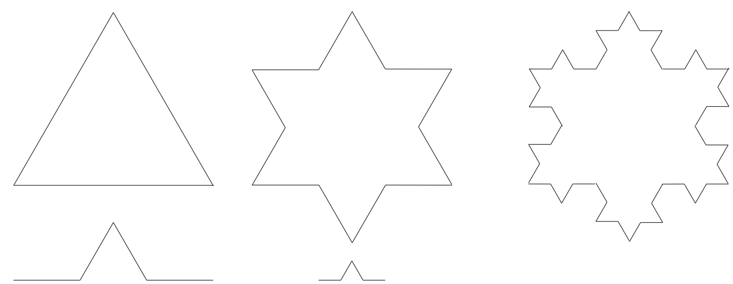


An object with a fractal boundary via repeated subdivision.

L-Systems (Lindenmayer-Systems)

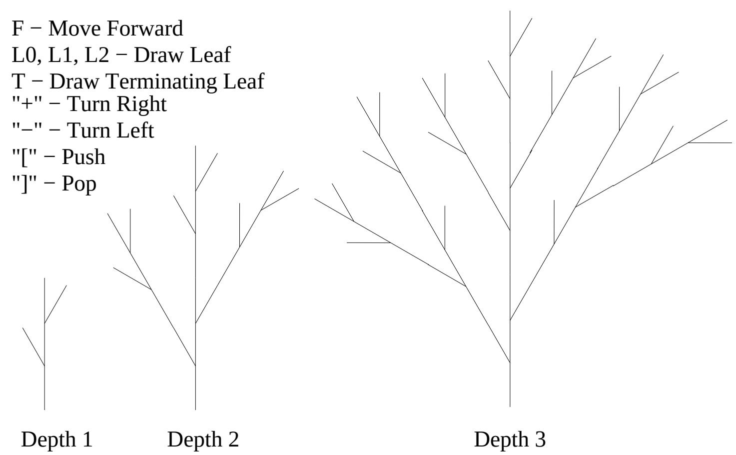
- Aristid Lindenmayer, a botanist, initially developed this as a mathematical theory for modeling plants
- Przemyslaw Prusinkiewicz (Dr. P.) fleshed this out for Graphics Modeling applications
- Central concept is of string rewriting, using productions or rewriting rules (e.g. F $\longrightarrow$ F + F - - F + F with all symbols +, - as characters not operators)
- Longer strings can be generated by repeated applications of the productions, starting from an axiom (e.g. F $\longrightarrow$ F + F - - F + F $\longrightarrow$ F + F - - F + F + F + F - - F + F - - F + F - - F + F + F + F - - F + F)
- See (http://mathforum.org/advanced/robertd/lsys2d.html) for other examples.

String Re-Writing and Turtle Graphics



- Turtle is a hypothetical drawing cursor on the screen or object coordinate system. Initially assume Turtle at origin (0,0) and facing UP.
- Interpret F as "Move turtle forward one unit and draw a line segment"
- Interpret - by "Turn counter-clockwise (ccw) by $\frac{\pi}{3}$"
- Interpret + "Turn clockwise (cw) by $\frac{\pi}{3}$"
- So then the string - F - - F - - F intepreted in Turtle graphics shall draw a triangle.
- Applying the production (or rule) F $\longrightarrow$ F + F - - F + F once to the axiom (- F - - F - - F) yields a Star.
- Iterated applications of this rule, yields the Koch snowflake fractal.

Constructing Trees

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Axiom: L0
1. L0 -> F [ - F L1 ] F [ + F L2 ] F L0  (center branch)
2. L1 -> F [ - F L1 ] F [ + F T  ] F L1  (left half of tree)
3. L2 -> F [ - F T  ] F [ + F L2 ] F L2  (right half of tree)
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- The Turtle can make wiggly paths, but not branching.
- For branching we use a Stack, and the L-system symbols [for Push, and] as Pop
- A stack can be implemented using OpenGl operators PushMatrix() and PopMatrix () or the Program Stack implicit in Recursion).

Transformation Modelling

- Model Turtle (position, direction, size) by a Matrix C
- We use OpenGL by loading C into MODELVIEW matrix.
- Assume Turtle initially at origin (0,0) and facing UP. This initial position is captured in C by the identity matrix
- Now we wish to find the transformation that moves Turtle to (50,100) and facing an angle $\frac{2\pi}{3}$
- This is obtained by the sequence of Modelling transformations T(50,100) R($\frac{\pi}{6}$) applied to C. Remember, the transformations need to be applied in the correct right2left order.

Using Recursion

- Use Recursion to replace PushMatrix (), PopMatrix() pairs with save and restore of the current matrix in the resolution of recursive calls by the Program Stack.
  PushMatrix()
    TurnRight() DrawLeaf(i-1)
  PopMatrix()
- DrawRightLeaf(i) Double[9]SavedMatrix;
    Copy(C,SavedMatrix); TurnRight() DrawLeaf(i-1) copy(SavedMatrix,C)
  PopMatrix()

Reading Assignment and News

Chapter 2 Exercises and Chapter 10 pages 497 - 520, of Recommended Text.

Please also track the News section of the Course Web Pages for the most recent Announcements related to this course.

(http://www.cs.utexas.edu/users/bajaj/graphics25/cs354/)