Automatic Parameterization of Rational Curves and Surfaces IV: Algebraic Space Curves

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For an irreducible algebraic space curve C that is implicitly defined as the intersection of two algebraic surfaces, $f(x, y, z) = 0$ and $g(x, y, z) = 0$, there always exists a birational correspondence between the points of C and the points of an irreducible plane curve P, whose genus is the same as that of C. Thus C is rational iff the genus of P is zero. Given an irreducible space curve $C = (f \cap g)$, with f and g not tangent along C, we present a method of obtaining a projected irreducible plane curve P together with birational maps between the points of P and C. Together with [4], this method yields an algorithm to compute the genus of C, and if the genus is zero, the rational parametric equations for C. As a biproduct, this method also yields the implicit and parametric equations of a rational surface S containing the space curve C.

The birational mappings of implicitly defined space curves find numerous applications in geometric modeling and computer graphics since they provide an efficient way of manipulating curves in space by processing curves in the plane. Additionally, having rational surfaces containing C yields a simple way of generating related families of rational space curves.

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1. INTRODUCTION

Consider an irreducible algebraic space curve C that is implicitly defined as the intersection of two algebraic surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$. We concern ourselves with space curves defined by two surfaces since they are of direct interest to applications in geometric modeling and computer graphics (e.g., [7]).

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Irreducible space curves in general, defined by more than two surfaces, are difficult to handle equationally, and one needs to resort to computationally intensive ideal-theoretic methods [9]. However general space curves is a topic with various unresolved issues of mathematical and computational interest and an area of important future research (e.g., [1]).

Now for an irreducible algebraic space curve C as above, there always exists a birational correspondence between the points of C and the points of an irreducible plane curve P whose genus is the same as that of C (see [19] and [20]). Birational correspondence between C and P means that the points of C can be given by rational functions of points of P and vice versa (i.e., a 1-to-1 mapping, except for a finite number of exceptional points of C and P).

In this paper we show how, given an irreducible space curve C, defined implicitly as the transversal intersection of two algebraic surfaces f and g (i.e., f and g are not tangent along C), one is able to construct the equation of a plane curve P and birational maps between the points of P and C. These birational maps, together with the method of computing the genus and rational parameterization of algebraic plane curves [4], then gives an algorithm to compute the genus of the space curve C, and if genus = 0, the rational parametric equations of C.

As a first attempt in constructing P, we may consider the projection of the space curve C along any of the coordinate axes yielding a plane curve whose points are in correspondence with the points of C. Projecting C along, say, the z axis, can be achieved by computing the Sylvester resultant of f and g, treating them as polynomials in z, yielding a single polynomial in x and y, the coefficients of f and g. The Sylvester resultant eliminates one affine variable, in this case z, from two polynomial equations (e.g., [15]). Efficient methods are known for computing this resultant for polynomials in any number of variables (e.g., [11]). The Sylvester resultant of f and g thus defines a plane algebraic curve P. However, this projected plane curve P in general is not in birational correspondence with the space curve C. For a chosen projection direction it is quite possible that most points of P may correspond to more than one point of C (i.e., a multiple covering of P by C), and hence the two curves are then not birationally related. See Figures 1, 2, and 4. This approach may be rectified, as explained in Section 2, by choosing a valid projection direction that yields a birationally related, projected plane curve P. See Figures 1 and 3. Further, the inverse rational map from the projected plane curve P to the original space curve C can also be efficiently constructed. Let the proper projected plane curve P be defined by the polynomial $h(\tilde{x}, \tilde{y})$. The map from C to P is linear and is given trivially by $\tilde{x} = x$ and $\tilde{y} = y$ (or related by a linear transformation as shown in Section 2). To construct the reverse rational map one only needs to compute $z = I(\tilde{x}, \tilde{y})$ where I is a rational function. We show in Section 3 how it is always possible to construct this rational function by use of a polynomial remainder sequence along a chosen valid projection direction. In fact the resultant is no more than the end result of a polynomial remainder sequence (see [6] and [14]).

The reverse rational map, $z = I(\tilde{x}, \tilde{y})$ where I is a rational function, is also the rational parametric equation of a rational surface containing the space curve C. Hence, constructing a birational mapping between space and plane curves that always exist, also yields an explicit rational surface containing the space curve.

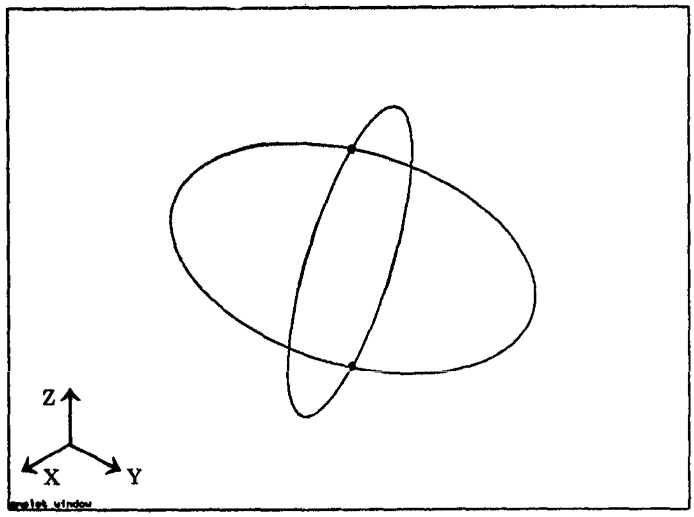

Fig. 1. Space curve C: $(f = z^2 + x^2 - 1 \cap g = z^2 + y^2 - 1)$.

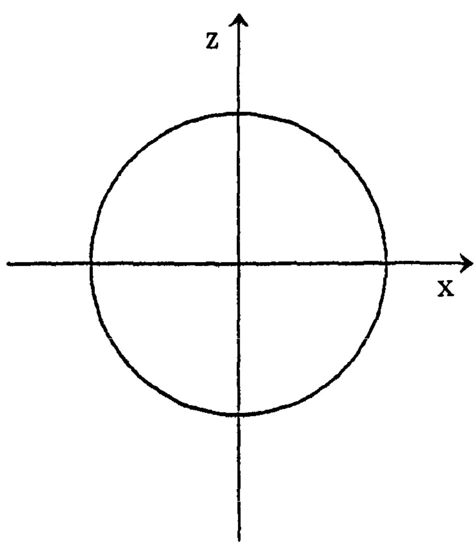

Fig. 2. Y axis projection P: $(x^2 + z^2 - 1)^2 = 0$.

By an explicit rational surface we mean one with a known or trivially derivable rational parameterization. For irreducible space curves C, a method of obtaining an explicit rational surface containing C is given (without proof) in [18]. Garrity and Warren [12] have also recently presented a general method of constructing birational maps between space curves and projected plane curves using differentiation arguments. The techniques presented in this paper differ in their choice of birational projections, as well as in their use of subresultant polynomial remainder sequences, to efficiently construct both a reverse rational map as well as an explicit rational surface containing C.

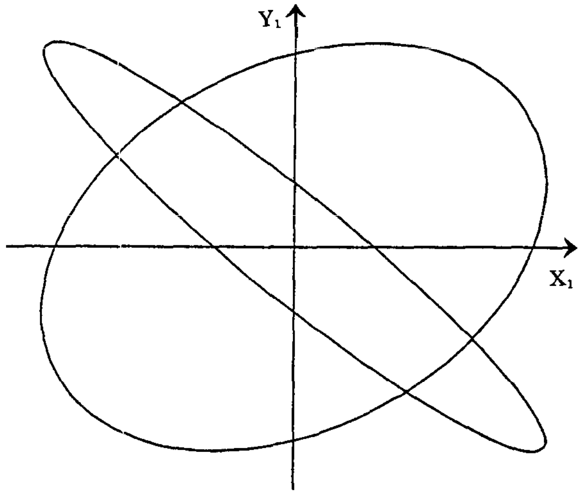

Fig. 3. Birationally projected P: $(8y_1^2 - 4x_1y_1 + 5x_1^2 - 9)(8y_1^2 + 12x_1y_1 + 5x_1^2 - 1) = 0$.

Note additionally, that knowing the rational parametric equations of a rational surface containing a space curve also yields a birational mapping between points on the space curve and a plane curve. Namely, if one of the two surfaces f or g defining the space curve C, or actually any known surface in $I(C) = I(f, g)$, the Ideal[1] of the curve C generated by f and g, is rational and with a known rational parameterization, then points on C are easily mapped to a single polynomial equation $h(s, t) = 0$ describing a plane curve P in the parametric plane $s - t$ of the rational surface. This mapping between the (x, y, z) points of C and the (s, t) points of P is birational with the reverse rational map from points on P to points on C being given by the parametric equations of the rational surface. For space curves C that have a quadric or a rational cubic surface in its Ideal, the plane curve P and the rational mapping from points on P to C are then easily constructed by using known techniques for parameterizing these rational surfaces (see [2], [3], and [17]).

The rest of this paper is structured as follows. Section 2 describes a method of choosing a valid direction of projection for the space curve C. This yields a projected plane curve P in birational correspondence to C. Using these results, Section 3 describes a method of constructing the reverse rational map between points on the plane curve P and points on C.

2. VALID PROJECTION DIRECTION

To find an appropriate axis of projection, the following general procedure may be adopted. Consider the general linear transformation $x = a_1x_1 + b_1y_1 + c_1z_1$, $y = a_2x_1 + b_2y_1 + c_2z_1$, and $z = a_3x_1 + b_3y_1 + c_3z_1$. On substituting into the equations of the two surfaces defining the space curve, we obtain the transformed

[1] $I(f, g) = \{h(x, y, z) \mid h = \alpha f + \beta g$ for any polynomials $\alpha(x, y, z)$ and $\beta(x, y, z)\}$.

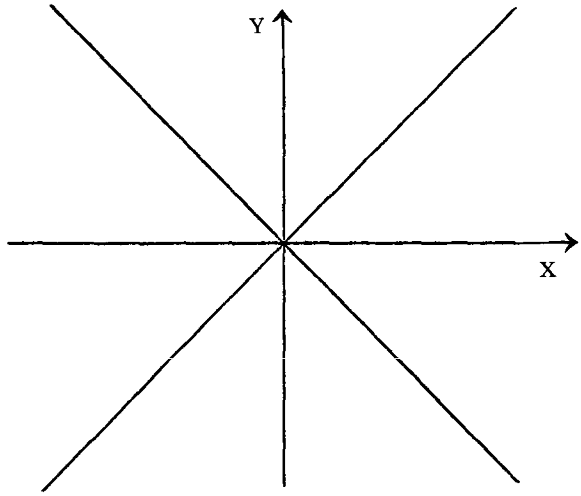

Fig. 4. Z axis projection P: $(y^2 - x^2)^2 = 0$.

equations $f_1(x_1, y_1, z_1) = 0$ and $g_1(x_1, y_1, z_1) = 0$. Next compute the $\mathrm{Res}_{z_1}(f_1, g_1)$, which is a polynomial $h(x_1, y_1)$ describing the projection along the Z axis of the space curve C onto the $z = 0$ plane.

Since C is irreducible and f and g are not tangent along C, the order of $h(x_1, y_1)$ is exactly equal to the projection degree (see [1] and [20]). By order of $h(x_1, y_1)$ we mean k if $h(x_1, y_1) = (g(x_1, y_1))^k$; see Figures 1 and 4. For a birational mapping we desire a projection degree equal to one; see Figures 1 and 3. Hence, we choose the coefficients of the linear transformation, a_i, b_i, and c_i such that

(1) the determinant of a_i, b_i, and c_i is nonzero
(2) the equation of the projected plane curve $h(x_1, y_1)$ is not a power of an irreducible polynomial. This can be achieved by ensuring that the discriminant $\mathrm{Res}_{x_1}(h_1, h_{x_1})$ is nonzero.

Note, a random choice of coefficients, or coefficients with sufficient bit length, would also work with probability 1 since the set of coefficients that make the determinant and $\mathrm{Res}_{x_1}(h_1, h_{x_1})$ equal to zero are restricted to the points of a lower dimensional hypersurface. See [16] where the notion of randomized computations with algebraic varieties is made precise. A suitable random choice of coefficients thus ensures that the projected irreducible plane curve given by $h(x_1, y_1)$ is in birational correspondence with the irreducible space curve and thus of the same genus. The parameterization methods of Abhyankar and Bajaj [4] for algebraic plane curves are now applicable and thereby yield a genus computation as well as an algorithm for rationally parameterizing the space curve.

Example 2.1 Let the given irreducible space curve C be defined as the transversal intersection of two equal radius circular cylinders $f = z^2 + x^2 - 1$ and $g = z^2 + y^2 - 1$. The curve is irreducible and consists of two intersecting ellipses in the planes $y = x$ and $y = -x$; see Figure 1. The resultant $\mathrm{Res}_x(f, g) = (y^2 - x^2)^2 = 0$ is the equation of the projected plane curve P and consists of two lines $y - x = 0$ and $y + x = 0$ both repeated twice; see Figure 3. The projection map between C and P here is thus two to one and not a valid projection. This occurs since the z axis, our axis of projection, is contained in the planes containing the curve C. Similarly, projecting along the y axis yields the plane curve P given by $\mathrm{Res}_y(f, g) = (x^2 + z^2 - 1)^2 = 0$, which is again not a valid projection. Here, two ellipses of the space curve C overlap in the projected plane curve P to again yield a two-to-one mapping between C and P. See Figure 2. A suitable linear transformation,

$$x = x_1 - 2 * z_1, \quad y = y_1 + z_1, \quad \text{and} \quad z = 2 * z_1$$

yields

$$f_1 = 8 * z_1^2 - 4 * x_1 * z_1 + x_1^2 - 1 \quad \text{and} \quad g_1 = 5 * z_1^2 + 2 * y_1 * z_1 + y_1^2 - 1,$$

whose resultant

$$\begin{aligned}\mathrm{Res}_{z_1}(f_1, g_1) &= 64 * y_1^4 + 64 * x_1 * y_1^3 + 32 * x_1^2 * y_1^2 - 80 * y_1^2 + 40 * x_1^3 * y_1 \\ &\quad - 104 * x_1 * y_1 + 25 * x_1^4 - 50 * x_1^2 + 9 \\ &= (8 * y_1^2 - 4 * x_1 * y_1 + 5 * x_1^2 - 9)(8 * y_1^2 + 12 * x_1 * y_1 + 5 * x_1^2 - 1),\end{aligned}$$

defines two distinct ellipses in the plane, that is, a projection map which is one to one. Note, the projected plane curves are ellipses of different shape since we are not restricted to using only orthogonal linear transformations.

3. CONSTRUCTING THE BIRATIONAL MAP

We choose a valid projection direction by using the method described in the earlier section. Without loss of generality let this direction be the Z axis. Let the surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ be of degrees m_1 and m_2 in z, respectively. Again, without loss of generality, assume $m_1 \geq m_2$. Let $F_1 = f(x, y, z)$ and $F_2 = g(x, y, z)$ be given by

$$\begin{aligned} F_1 &= f_0 z^{m_1} + f_1 z^{m_1 - 1} + \cdots + f_{m_1 - 1} z + f_{m_1} \\ F_2 &= g_0 z^{m_2} + g_1 z^{m_2 - 1} + \cdots + g_{m_2 - 1} z + g_{m_2} \end{aligned} \tag{1}$$

where f_j, $(j = 0 \cdots m_1)$ and g_k, $(k = 0 \cdots m_2)$ are polynomials in x, y. Then there exist polynomials $F_{i+2}(x, y, z)$ for $i = 1 \cdots r$ such that $B_i F_{i+2} = A_i F_i - Q_i F_{i+1}$, where m_{i+2}, the degree of z in F_{i+2}, is less than m_{i+1}, the degree of z in F_{i+1}, for certain polynomials $A_i(x, y)$, $B_i(x, y)$, and $Q_i(x, y, z)$. The sequence of polynomials F_i, $i = 1, 2, \ldots, k$ is naturally known as a generalized polynomial remainder sequence (PRS) and can be computed in different ways, as we now describe.

Let $\mathrm{lc}(F)$ denote the leading coefficient of a polynomial $F(x, y, z)$, viewed as a polynomial in z, (i.e., a coefficient of term with highest z degree). Further let c_i

denote $\mathrm{lc}(F_i)$. To compute F_{i+2} from F_i and F_{i+1} we first begin with $R_i^0 = F_i$ and then,

$$\begin{aligned}
&\textbf{for} \quad \mathbf{k} = 1, \ldots, m_i - m_{i+1} + 1 \\
&\quad \textbf{if } \mathrm{lc}(R_i^{k-1}) = 0 \\
&\quad \textbf{then } R_i^k = R_i^{k-1} \\
&\quad \textbf{else } R_i^k = c_{i+1} R_i^{k-1} - z^{m_i - m_{i+1}+1-k} \mathrm{lc}(R_i^{k-1}) F_{i+1}
\end{aligned} \tag{2}$$

The polynomial $R_i^{m_i - m_{i+1}+1}$ is known as the sparse pseudoremainder of F_i and F_{i+1}. Using Collin's reduced PRS method [10], one constructs the polynomial $F_{i+2} = R_i^{m_i - m_{i+1}+1}/d_{i-1}$ where $d_0 = 1$ and $d_i = c_{i+1}^{m_i - m_{i+1}+1}$. Using Brown's subresultant PRS scheme [8], one constructs the polynomial

$$F_{i+2} = (-1)^{m_i - m_{i+1}+1} (R_i^{m_i - m_{i+1}+1} / c_i E_{m_i}^{m_i - m_{i+1}}),$$

where $E_{m_1} = 1$ and $E_{m_i} = c_i^{m_{i-1} - m_i} / E_{m_{i-1}}^{m_{i-1} - m_i - 1}$. As shown by Loos [14], both the above methods, as well as others, follow naturally from the subresultant theorem of Habicht [13].

Thus starting with polynomials F_1 and F_2, one constructs the polynomial remainder sequence, $F_1, F_2, F_3, \ldots F_i, \ldots F_r$ such that m_r = the z degree of $F = 0$ (i.e., F_r being independent of z). We choose the subresultant PRS scheme for its computational superiority and also because each $F_i = S_{m_{i-1}-1}$, $1 \le i \le r$, where S_k is the kth subresultant of F_1 and F_2 (see [8], [10], and [13]).

Now for any $i = 1, \ldots, r-2$, if F_i and F_{i+1} are of degree greater than two and F_{i+2} is independent of z, then the Z axis is not a valid projection direction. This may be seen as follows. Since the Z axis was chosen as a valid projection direction, the $\mathrm{Res}_z[f(x, y, z), g(x, y, z)] = \mathrm{Res}_z[F_1, F_2] = S_0$ is nonzero and not a multiple of some irreducible polynomial. This holds for any two surfaces F_i and F_{i+1} in the PRS, all elements in the Ideal of C, generated by f and g. If any of the elements F_i of the PRS are multiples of some irreducible polynomial then so would the resultant, which is impossible. To complete the argument, it remains to see that by induction, if F_{i-1} and F_i are of say degree three and two respectively and F_{i+1} is independent of z, then the $\mathrm{Res}_z(F_{i-1}, F_i)$ is equal to some $h^2(x, y)$, which is impossible.

Hence in the PRS, for a valid projection axis, there exists an element that is linear in z, that is, $F_{i-1} = z\Phi_1(x, y) - \Phi_2(x, y) = 0$. Thus on computing the PRS and obtaining F_{r-1}, one is able to construct the required inverse map, $z = \Phi_2(x, y)/\Phi_1(x, y)$, which is also a rational surface containing the space curve. The rational parameterization of this rational surface is trivially given by $x = s$, $y = t$, and $z = \Phi_2(s, t)/\Phi_1(x, t)$. Note that the two coefficients Φ_1 and Φ_2 of F_{r-1} cannot have a common factor divisible by the resultant F_r, for then F_r, which is the pseudoremainder of F_{r-2} and F_{r-1}, would again contain a factor raised to a certain power.

Example 3.1 Let the given irreducible space curve C be defined as the transversal intersection of $f = F_1 = z^3 + 4 * z + y^2$ and $g = F_2 = z^2 + 2 * z + x^2$; see Figure 5. Computing the subresultant PRS yields $F_3 = (8 - x^2) * z + (2 * x^2 + y^2)$ and $F_4 = y^4 + 6 * x^2 * y^2 - 16 * y^2 + x^6 - 8 * x^4 + 32 * x^2$. The

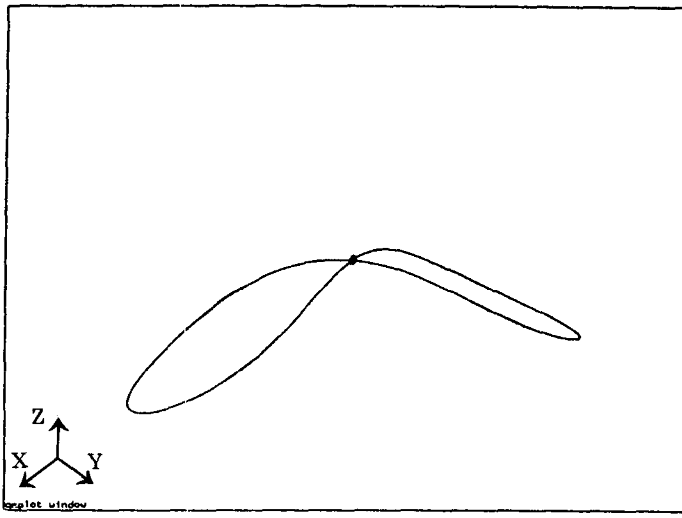

Fig. 5. Space curve C: ($f = z^3 + 4z + y^2 \cap g = z^2 + 2z + x^2$).

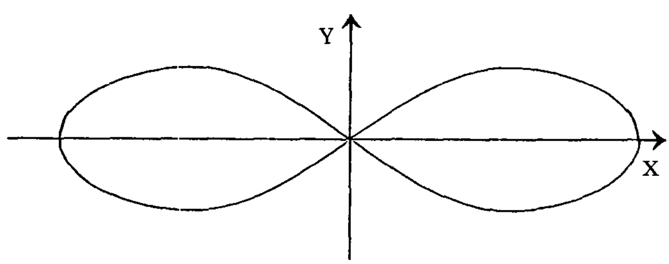

Fig. 6. Birationally projected P: $y^4 + 6x^2y^2 - 16y^2 + x^6 - 8x^4 + 32x^2 = 0$.

resultant $\mathrm{Res}_z(f, g) = F_4 = 0$ is the equation of the projected plane curve P and is square free; see Figure 6. The rational surface containing the curve C is $F_3 = 0$ or alternatively given by ($x = s$, $y = t$, and $z = (2 * s^2 + t^2)/(8 - t^2)$).

Example 3.2 Let the given irreducible space curve C be defined as in Example 2.1. For the given $f = F_1 = z^2 + x^2 - 1$ and $g = F_2 = z^2 + y^2 - 1$ computing the subresultant PRS yields $F_3 = 0 * z + (-x^2 + y^2)$, and hence the z axis is not a valid projection direction. For the transformed space, under the linear transformation

$$x = x_1 - 2 * z_1, \qquad y = y_1 + z_1, \qquad z = 2 * z_1,$$

$$f_1 = F_1 = 8 * z_1^2 - 4 * x_1 * z_1 + x_1^2 - 1,$$

and

$$g_1 = F_2 = 5 * z_1^2 + 2 * y_1 * z_1 + y_1^2 - 1.$$

The subresultant PRS yields

$$F_3 = (20 * x + 16 * y) * z + (-5 * x^2 + 8 * y^2 - 3)$$

and

$$\begin{aligned} F_4 &= 64 * y_1^4 + 64 * x_1 * y_1^3 + 32 * x_1^2 * y_1^2 - 80 * y_1^2 + 40 * x_1^3 * y_1 \\ &\quad - 104 * x_1 * y_1 + 25 * x_1^4 - 50 * x_1^2 + 9 \\ &= (8 * y_1^2 - 4 * x_1 * y_1 + 5 * x_1^2 - 9)(8 * y_1^2 + 12 * x_1 * y_1 + 5 * x_1^2 - 1) \end{aligned}$$

as the resultant of f_1 and g_1 as before. The rational surface containing the curve C is $F_3 = 0$, or alternatively given by ($x = s$, $y = t$, and $z = (-5 * s^2 + 8 * t^2 - 3)/(20 * s + 16 * t)$).

4. CONCLUSION

The assumption that the above space curve C is irreducible stemmed from our primary motivation of parameterizing implicitly defined space curves. However, the irreducibility assumption is not necessary for the methods of Sections 2 and 3, and the algorithms presented there for constructing birational maps apply directly for reducible space curves as well. One chooses a valid projection direction as before by making the discriminant of the projected plane curve P to be nonzero, which also ensures that two or more space curve components do not get projected over the same plane curve component. (See again Example 2.1 of Section 2).

One limitation of our method, however, is the assumption of nontangency of the surfaces f and g meeting along the space curve C. This has recently been removed by the method of Garrity and Warren [12] using bivariate polynomial GCD and division computations to achieve squarefree polynomials for the projected plane curve P. However, the problem of finding computationally efficient algorithms to construct birational maps for space curves, defined implicitly as the intersection of two parametric surfaces, remains open.

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