Hermite Interpolation using Real Algebraic Surfaces

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Abstract

We present a simple characterization of the lowest degree, implicitly defined, real algebraic surfaces, which smoothly contain any given number of points and algebraic space curves, of arbitrary degree. The characterization is constructive, yielding efficient algorithms for generating families of such algebraic surfaces. Smooth containment of space curves yields C^1-continuous surface fitting, and is a generalization of standard Hermite interpolation applied to fitting curves through point data, equating derivatives at those points. We deal with the containment and matching of "normals" (vectors orthogonal to tangents), possibly varying along the entire span of the space curves. Such Hermite interpolated surfaces prove useful as "blending" or "joining" surfaces for solid models as well as "fleshing" surfaces for curved wireframe models.

1 Introduction

Importance: While developing a geometric modeling system for the construction of accurate computer models of solid physical objects [1], we have designed a technique of automatically generating real interpolation surfaces of low degree, which yields a piecewise, tangent-plane-continuous mesh of algebraic surface patches. Modeled physical objects with algebraic surface patches of the lowest degree, lends itself to faster computations in geometric design operations as well as in tasks such as computer graphics display, animation, and various physical simulations.

Geometric Coverage: We focus on the use of low degree, implicitly defined, algebraic surfaces in three dimensional space $\mathbb{R}^3$. A real algebraic surface S in $\mathbb{R}^3$ is implicitly defined by a single polynomial equation $f(x, y, z) = 0$, where coefficients of f are over the real numbers $\mathbb{R}$. A real algebraic space curve can be defined by the intersection of two real algebraic surfaces and implicitly represented as a pair of polynomial equations ($f_1(x,y,z) = 0$ and $f_2(x,y,z) = 0$) with coefficients again over the real numbers $\mathbb{R}$. In modeling the boundary of physical objects it suffices to consider only space curves defined by the intersection of two algebraic surfaces. Space curves in general are defined by the intersection of several surfaces. A *rational* algebraic space curve can also be represented by the triple $(x = G_1(s), y = G_2(s), z = G_3(s))$, where G_1, G_2 and G_3 are rational functions in s. Whenever we consider the special case of a rational space curve, we assume that the curve is smooth and only singly defined under the parameterization map, i.e., each triple of values for (x, y, z), corresponds to a single value of s.

Why algebraic surfaces ? Manipulating polynomials, as opposed to arbitrary analytic functions, is computationally more efficient. Furthermore algebraic surfaces provide enough generality to accurately model almost all complicated rigid objects. Also as we show here, algebraic curves and surfaces lend themselves very naturally to the difficult problem of Hermite interpolation.

Why implicit representations ? Most prior approaches to interpolation and surface fitting, have focused on the parametric representation of surfaces [3,12,16]. Contrary to major opinion and as we exhibit here, implicitly defined surfaces are also very appropriate for interpolation. Additionally, while all algebraic surfaces can be represented implicitly, only a subset of them have the alternate parametric representation, with x, y and z given explicitly as rational functions of two parameters. Furthermore, implicit algebraic curves and surfaces have compact storage representations and form a class which is closed under most common oper-

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ations required by a geometric modeling system.

The Problem: Construct a real algebraic surface S, which *smoothly* interpolates a collection of k points $\mathbf{p}_i$ in $\mathbb{R}^3$ with associated fixed "normal" unit vectors $\mathbf{m}_i$, and l given space curves C_j in $\mathbb{R}^3$ also with associated "normal" unit vectors $\mathbf{n}_j$, varying along the entire span of the curves, $(i = 1 \ldots k,\ j = 1 \ldots l)$. Both points and space curves have an infinity of potential "normal" vector directions. While for points the $\mathbf{m}_i$ may be chosen arbitrarily, for space curves C_j, the varying unit vectors $\mathbf{n}_j$ are chosen to be always orthogonal to the tangent vector $\mathbf{t}_j$, that is, $\mathbf{t}_j \cdot \mathbf{n}_j = 0$, along the entire curve. Our emphasis being algebraic space curves, the variance of the curves "normals" are restricted to univariate polynomials of some degree. Also, we assume that any of the vectors $\mathbf{m}_i$ and $\mathbf{n}_j$ are never identically zero, a phenomenon that occurs at point and curve singularities. By *smoothly* interpolates we shall mean that S contains each of the points and curves and furthermore has its gradient in the same direction as the "normal" vectors $\mathbf{m}_i$ and $\mathbf{n}_j$. This is a natural generalization of Hermite interpolation, applied to fitting curves through point data, and equating derivatives at those points. As we shall see later, the choice of the associated "normal" direction, in each case is dictated by the use of the Hermite interpolated surface, (eg, in "blending" or "joining" or "fleshing").

Related Work: Sarraga in [12] presents techniques for constructing a C^1-continuous surface of rectangular Bézier (parametric) surface patches, interpolating a net of cubic Bézier curves. Other approaches to parametric surface fitting and transfinite interpolation are also mentioned in that paper, as well as in [16]. An excellent exposition of exact and least squares fitting of algebraic surfaces through given data points, is presented in [10]. Meshing of given algebraic surface patches using control techniques of joining Bézier polyhedrons is shown in [13]. Surface blending consisting of "rounding" and "filleting" surfaces (smoothing the intersection of two primary surfaces), a special case of Hermite interpolation, has been considered for polyhedral models in [4] and for algebraic surfaces in [5,6,8,9,11,14,15,16].

Results: We show in Sections 3, 4 and 5 that the problem of generalized Hermite interpolation of points and curves with algebraic surfaces, reduces to solving systems of linear equations, albeit at times with symbolic coefficients. In particular for an algebraic surface of degree n, to *smoothly contain* k points and l space curves of degree d with assigned "normal" directions, varying as a polynomial of degree m, the number of linear equations to be satisfied is $3k + (2n + m - 1)dl + 2l$. This number reduces to $3k + (2n - 1)dl + ml + 2l$ when all the space curves and "normals" are represented parametrically. Since the number of independent coefficients (unknowns) of a general algebraic surface of degree n is $\binom{n+3}{3} - 1$, the number of linear equations stated above, yields both necessary and sufficiency conditions on Hermite interpolated algebraic surfaces, for a variety of point and curve data configurations. As applications of this simple vector space characterization of Hermite interpolated algebraic surfaces, we show, in section *5.*, for example, that:

- Two space lines with constant-direction normals can be Hermite interpolated with a real quadric if and only if the lines are parallel or intersect at a point, and the normals are not orthogonal to the plane containing them. The real quadric is a "cylinder" when the lines are parallel and a "cone" when the lines intersect.

- Two skewed lines with constant-direction normals cannot be Hermite interpolated with real quadrics. The only real quadratic surface which satisfies both containment and tangency conditions reduces into two planes.

- The minimum degree of a real algebraic surface, which Hermite interpolates two lines in space, one with a constant direction normal, the other with a linearly varying normal is three.

- Two lines with linearly varying normals can be Hermite interplated by a quadric in only some special cases. In general, a surface of at least degree three is needed. When real quadric surface interpolation is possible, the real quadric is either a hyperboloid of one sheet (the two lines may be parallel, intersecting, or skewed) or a hyperbolic paraboloid (the two lines can only be intersecting or skewed).

Lines in space with constant-direction normals, occur naturally as edges of polyhedra, with the Hermite interpolating surfaces being used to "smooth" planar faces containing those edges. Lines with linearly-varying normals occur on real quadric and cubic surfaces. Similar results to the ones above, are also derived in sections*5.* and *6.*, for Hermite interpolation of conics and cubics in space. Since these rational curves lie on quadrics, cubic surfaces and higher degree algebraic surfaces, our method gives a powerful way of automatically, generating low degree "blending" and "joining" and "fleshing" surfaces with tangent continuity at intersections.

2 Preliminaries

For any multivariate polynomial f, partial derivatives are written by subscripting, for example, $f_x = \partial f / \partial x$, $f_{xy} = \partial^2 f / (\partial x \partial y)$, and so on. Since we consider algebraic curves and surfaces, we have $f_{xy} = f_{yx}$ etc. Vectors and vector functions are denoted by bold letters. The *inner product* of vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted $\mathbf{a} \cdot \mathbf{b}$. The *length* of the vector $\mathbf{a}$ is $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

The *gradient* of $f(x,y,z)$ is the vector $\nabla f = (f_x, f_y, f_z)$. A point $p = (x_0, y_0, z_0)$ is a *simple* point of f if the gradient of f at p is not null; otherwise the point is *singular*. An algebraic surface is non-singular or smooth if all its points are simple.

Definition 2.1 *Let* $\mathbf{p} = (a,b,c)$ *be a point with an associated "normal"* $\mathbf{m} = (m_x, m_y, m_z)$ *in* $\mathbb{R}^3$. *An algebraic surface* $S : f(x,y,z) = 0$ *is said to* smoothly contain $\mathbf{p}$ *if*
(1) $f(\mathbf{p}) = f(a,b,c) = 0$, **(containment condition)**
and
(2) $\nabla f(\mathbf{p})$ *is not zero and* $\nabla f(\mathbf{p}) = \alpha\mathbf{m}$, *for some nonzero* α. **(tangency condition)**

Definition 2.2 *Let* C *be an algebraic space curve with an associated varying "normal"* $\mathbf{n}(x,y,z) = (n_x(x,y,z), n_y(x,y,z), n_z(x,y,z))$, *defined for all points on* C. *An algebraic surface* $S : f(x,y,z) = 0$ *is said to* smoothly contain C *if*
(1) $f(\mathbf{p}) = 0$ *for all points* $\mathbf{p}$ *of* C. **(containment condition)**
and
(2) $\nabla f(\mathbf{p})$ *is not identically zero and* $\nabla f(\mathbf{p}) = \alpha\mathbf{n}(\mathbf{p})$, *for some nonzero* α *and for all points* $\mathbf{p}$ *of* C. **(tangency condition)**

Definition 2.3 *An algebraic surface* $S : f(x,y,z) = 0$ *is said to Hermite interpolate a given collection of data points with associated "normals", and data curves with associated "normals", if* S smoothly contains *all the data points and curves.*

The following is one form of Bezout's theorem (the oldest theorem of algebraic geometry).

Theorem 2.1 *An algebraic curve* C *of degree* d *intersects an algebraic surface* S *of degree* n *in at most* nd *points, or else it must intersect it infinitely often, that is, a component of* C *must lie entirely on* S.

3 Interpolation of Points

3.1 Containment

There exist applications in object reconstruction in geometric design, when there is need to construct a surface which interpolates a given set of data points. From the containment condition of definition 2.1 it directly follows that any algebraic surface $S : f(x,y,z) = 0$, whose coefficients satisfy the linear equation $f(\mathbf{p}) = 0$ will contain the point $\mathbf{p}$. For a set of k data points this yields k linear equations. For an algebraic surface of degree n, having $K = \binom{n+3}{3} - 1$ independent coefficients, various types of exact fits can be obtained by choosing the smallest n such that $K \geq r$, where r, $(\leq k)$ is the rank of the system of k linear equations. Details of methods for constructing such real algebraic surfaces can be found in [10].

3.2 Containment with Tangency

Quite often one also needs a low degree algebraic surface which not only contains a set of data points but is also tangent to a prespecified plane at each of those points. A point $\mathbf{p} = (a,b,c)$ with a "normal" $\mathbf{m} = (m_x, m_y, m_z)$ determines a unique plane $P : m_x x + m_y y + m_z z - (m_x a + m_y b + m_c c) = 0$, at the point $\mathbf{p}$. An algebraic surface $S : f(x,y,z) = 0$ of degree n that Hermite interpolates a point $\mathbf{p}$, can be computed as follows:

1. **(containment condition)** For point $\mathbf{p}$ set up the linear equation $f(\mathbf{p}) = 0$ in the unknown coefficients of S.
2. **(tangency condition)** One of the following cases is selected.
   (a) If $m_x \neq 0$, use the equations $m_x f_y(\mathbf{p}) - m_y f_x(\mathbf{p}) = 0$ and $m_x f_z(\mathbf{p}) - m_z f_x(\mathbf{p}) = 0$.
   (b) If $m_y \neq 0$, use the equations $m_y f_x(\mathbf{p}) - m_x f_y(\mathbf{p}) = 0$ and $m_y f_z(\mathbf{p}) - m_z f_y(\mathbf{p}) = 0$.
   (c) If $m_z \neq 0$, use the equations $m_z f_x(\mathbf{p}) - m_x f_z(\mathbf{p}) = 0$ and $m_y f_z(\mathbf{p}) - m_z f_y(\mathbf{p}) = 0$.
3. We also ensure that the coefficients of $f(x,y,z) = 0$ satisfying the above three linear equations, additionally satisfy the linear constraints $\nabla f(\mathbf{p}) \neq 0$, since non-tangency at $\mathbf{p}$ may occur if S turns out to be singular at $\mathbf{p}$.

The proof of correctness of the above algorithm follows from the following lemma.

Lemma 3.1 *The equations of the above algorithm satisfy defintion 2.1 of point containment and tangency.*

Proof : The first linear equation $f(\mathbf{p}) = 0$ satisfies containment by definition. We now show that the remaining equations satisfy $\nabla f(\mathbf{p}) = \alpha \cdot \mathbf{m}$ for a nonzero α. Since $\mathbf{m}$ is never taken to be the $(0,0,0)$ vector, without loss of generality we may assume that $m_x \neq 0$ in step *2.* above. Other cases of $m_y \neq 0$ or $m_z \neq 0$ can be handled symmetrically. Now let $\alpha = \frac{f_x}{m_x}$, assuming $m_x \neq 0$. Then $f_x = \alpha \cdot m_x$ and susbtituting it in the selected linear equation $m_x f_y - m_y f_x = 0$ yields $f_y = \alpha \cdot m_y$ and substituing it again in the other selected linear equation $m_x f_z - m_z f_x = 0$ yields $f_z = \alpha \cdot m_z$. Hence $\nabla f(\mathbf{p}) = \alpha \cdot \mathbf{m}$. Finally, note that $f_x = 0$ for $m_x \neq 0$, in the selected linear equations of step *2 (a).*, would cause $\nabla f(\mathbf{p}) = 0$, which we ensured would not happen in step *3* of the algorithm. Hence $f_x \neq 0$ and so $\alpha \neq 0$ and the lemma is proved. ♠

4 Interpolation of Curves

The varying "normal" associated with a space curve C can be defined implicitly by the triple $\mathbf{n}(x,y,z) =$

$(n_x(x,y,z), n_y(x,y,z), n_z(x,y,z))$ where n_x, n_y and n_z are polynomials of maximum degree m and defined only for all points $\mathbf{p} = (x,y,z)$ along the curve C. For the special case of a rational curve, as defined earlier, and which we shall treat separately in sections *4.1.2* and *4.2.2*, the varying "normals" can also be defined parametrically as $\mathbf{n}(s) = (n_x(s), n_y(s), n_z(s))$, with n_x, n_y and n_z now rational functions in s.

4.1 Containment

4.1.1 Algebraic Curves: Implicit Definition

Let $\mathbf{C} : (f_1(x,y,z) = 0, f_2(x,y,z) = 0)$ implicitly define an irreducible algebraic space curve of degree d. The irreducibility of the curve is not really a restriction, since reducible curves can be handled similarly by treating each irreducible component in turn. The situation is slightly more complicated if in the real setting, we may wish to achieve separate containment of each real component of an irreducible curve. We defer a solution to this problem, and for the time being consider it reduced to the problem of choosing appropriate clipping surfaces to isolate that real component, after the interpolated surface is computed. Note for parametrically defined curves, this problem does not arise.

An interpolating surface $S : f(x,y,z) = 0$ of degree n for containment of C, is then computed as follows:

1. Choose a set L_c of $nd+1$ points on $\mathbf{C}$, $L_c = \{\mathbf{p}_i = (x_i, y_i, z_i) | i = 1, \cdots, nd+1\}$[1]. The set L_c may be computed, for example, by tracing the intersection of $f_1 = f_2 = 0$, see for e.g., [2].

2. Next, set up $nd+1$ homogenous linear equations $f(\mathbf{p}_i) = 0$, for $\mathbf{p}_i \epsilon L_c$. Any nontrivial solution of this linear system will represent an algebraic surface which interpolates the entire curve $\mathbf{C}$.

The proof of correctness of the above algorithm is captured in the following Lemma.

Lemma 4.1 *To satisfy the containment condition of an algebraic curve C of degree d by an algebraic surface S of degree n, it suffices to satisfy the containment condition of $nd+1$ points of C by S.*

Proof: This is essentially a restatement of Bezout's theorem of section 2. By making S contain $nd+1$ points of $\mathbf{C}$, ensures that S must intersect C infinitely often and since C is irreducible, S must contain the entire curve. ♠

Remember $S : f(x,y,z) = 0$ of degree n has $K = \binom{n+3}{3} - 1$ independent coefficient unknowns. Let r be the rank of the system of $nd+1$ linear equations. There are non-trivial solutions to this homogeneous system if and only if $K > r$ and a unique non-trivial solution when $K = r$. Hence, again an interpolating surface can be obtained by choosing the smallest n such that $K \geq r$.

[1] Thus, alternatively, an algebraic curve may be given as a list of points.

4.1.2 Rational Curves : Parametric Definition

When a curve is given in rational parametric form, its equations can be used directly to produce a linear system for interpolation, instead of first computing $nd+1$ points on the curve. Let $\mathbf{C} : (x = G_1(t), y = G_2(t), z = G_3(t))$ be a rational curve of degree d. An interpolating surface $S : f(x,y,z) = 0$ of degree n which contains C is computed as follows:

1. Substitute $(x = G_1(t), y = G_2(t), z = G_3(t))$ into the equation $f(x,y,z) = 0$.

2. Simplify and rationalize to obtain $Q(t) = 0$, where Q is a polynomial in t, of degree at most nd, and with coefficients which are linear expressions in the coefficients of f. For Q to be identically zero, each of its coefficents must be zero, and hence we obtain a system of at most $nd+1$ linear equations, where the unknowns are the coefficients of f. Any nontrivial solution of this linear system will represent a surface S which interpolates $\mathbf{C}$.

The proof of correctness of the algorithm follows from the lemma below.

Lemma 4.2 *The containment condition is satisfied by step 2. of the above algorithm*

Proof: We omit this here and refer the reader to the full paper. ♠

4.2 Containment with Tangency

In order to Hermite interpolate an algebraic curve C with associated "normals" $\mathbf{n}$ by an algebraic surface S, we need to again solve a homogenous linear system, whose equations stem from both the *containment condition* and the *tangency conditions* of definition 2.2.

4.2.1 Algebraic Curves with Normals: Implicit Definition

As before, let $C : (f_1(x,y,z) = 0, f_2(x,y,z) = 0)$ implicitly define an irreducible algebraic space curve of degree d, together with associated "normals" defined implicitly by the triple $\mathbf{n}(x,y,z) = (n_x(x,y,z), n_y(x,y,z), n_z(x,y,z))$ where n_x, n_y and n_z are polynomials of maximum degree m and defined for all points $\mathbf{p} = (x,y,z)$ along the curve C. A Hermite interpolating surface $S : f(x,y,z) = 0$ of degree n which smoothly contains C is then computed as follows:

1. Choose a set L_c of $(n+m-1)d+1$ points on C, $L_c = \{\mathbf{p}_i = (x_i, y_i, z_i) | i = 1, \cdots, (n+m-1)d+1\}$. The set L_c may be computed, as before, by tracing the intersection of $f_1 = f_2 = 0$, see for e.g., [2].

2. Construct the list L_t of $(n+m-1)d+1$ point-normal pairs on C, $L_t = \{[(x_i, y_i, z_i), (n_{xi}, n_{yi}, n_{zi})] | i = 1, \cdots, (n+m-1)d+1\}$[2], where $(n_{xi}, n_{yi}, n_{zi}) = \mathbf{n}(\mathbf{p}_i)$ for $\mathbf{p}_i \epsilon L_c$.

3. **(containment condition)** Next, set up $nd+1$ homogenous linear equations $f(\mathbf{p}_i) = 0$, for $\mathbf{p}_i \epsilon L_c$ and $i = 1, \cdots, nd+1$.

4. **(tangency condition)**
   (a) Compute $\mathbf{t}(x,y,z) = \nabla f_1(x,y,z) \times \nabla f_2(x,y,z)$. Note $\mathbf{t} = (t_x, t_y, t_z)$ is the tangent vector to C.
   (b) One of the following cases is selected.
      i. If $t_x \neq 0$, use the equation $f_y \cdot n_z - n_y \cdot f_z = 0$.
      ii. If $t_y \neq 0$, use the equation $f_x \cdot n_z - n_x \cdot f_z = 0$.
      iii. If $t_z \neq 0$, use the equation $f_x \cdot n_y - n_x \cdot f_y = 0$.

   Substitute each point-normal pair in L_t into the above selected equation to yield additionally $(n+m-1)d+1$ linear equations in the coefficients of the $f(x,y,z)$.

5. In total we obtain a homogeneous system of $(2n+m-1)d+2$ linear equations. Any non-trivial solution of the linear system, for which additionally ∇f is not identically zero for all points of C, (that is, the surface S is nonsingular at all points along the curve C), will represent a surface which Hermite interpolates C.

The proof of correctness of the above algorithm follows from Lemma 4.1 and the following lemma, which shows why the selected equation of step *4.(b)* evaluated at $(n+m-1)d+1$ point-normal pairs, are sufficient.

Lemma 4.3 *To satisfy the tangency condition of an algebraic curve C of degree d with "normal" $\mathbf{n}$ of degree m, by an algebraic surface S of degree n, it suffices to satisfy the tangency condition at $(n+m-1)d+1$ points of C by S as in step 4. of the above algorithm.*

Proof : In step *4.(b)* above, assume without loss of generality that $t_x \neq 0$. Then the selected equation

$$f_y \cdot n_z - n_y \cdot f_z = 0 \tag{1}$$

We first show that even though equation (1) is evaluated at only $(n+m-1)d+1$ points of C in step *4.(b)* above, it holds for all points on C. Equation 1 defines an algebraic surface T of degree $(n+m-1)$ which intersects C of degree d at $(n+m-1)d$ real points. Invoking Bezout's theorem, and from the irreducibility of C, it follows that C must lie entirely on the surface T. Hence equation (1) is valid along the entire curve C.

We now show that step 4. of the above algorithm, satisfies the tangency condition as specified in definition 2.2. Since $\mathbf{t}$ of step *4.(a)* is a tangent vector at all points of C, and the surface $S : f = 0$ contains C, the gradient vector ∇f is orthogonal to $\mathbf{t}$, which yields the equation :

$$f_x \cdot t_x + f_y \cdot t_y + f_z \cdot t_z = 0 \tag{2}$$

valid for all points of C. Next, from the definition of a "normal" of a space curve,

$$n_x \cdot t_x + n_y \cdot t_y + n_z \cdot t_z = 0 \tag{3}$$

valid for all points of C. Now it is impossible that both $n_y(x,y,z)$ and $n_z(x,y,z)$ are identically zero along C, since if they were then equation (3) would imply that $n_x \cdot t_x = 0$, and as we had assumed that $t_x \neq 0$, would in turn imply that also $n_x = 0$ along C, which would contradict the earlier assumption that $\mathbf{n}$ is not identically zero. Hence, at least, one of n_y and n_z must also be nonzero. Without loss of generality, let $n_y \neq 0$. Also, let $\alpha(x,y,z) = \frac{f_y}{n_y}$. Then,

$$f_y = \alpha \cdot n_y \tag{4}$$

and substituting into equation (1) yields

$$f_z = \alpha \cdot n_z \tag{5}$$

for all points on C. From equations (2), (4) and (5) we obtain,

$$f_x \cdot t_x + \alpha \cdot n_y \cdot t_y + \alpha \cdot n_z \cdot t_z = 0 \tag{6}$$

By multiplying α to equation (3) and subtracting equation (6) from it, we obtain

$$f_x \cdot t_x = \alpha \cdot n_x \cdot t_x \tag{7}$$

and since $t_x \neq 0$, finally obtain

$$f_x = \alpha \cdot n_x \tag{8}$$

valid at all points of C. Hence equations (4), (5), and (8) together imply that $\nabla f(x,y,z) = \alpha \cdot \mathbf{n}$ for all points C and some nonzero α[3]. Hence, the tangency condition of definition 2.2 is met. ♠

[2] Thus, alternatively, an algebraic curve $\mathbf{C}$ and its associated "normals" $\mathbf{n}$ may (either or both) be given as a list of points or point-normal pairs.

[3] From equation (6) we see that $\alpha(x,y,z)$ must not be identically zero along C, for otherwise, $\nabla f = (0,0,0)$ for points along C and would contradict the fact that we chose a non-trivial solution for the surface $S : f = 0$ which was nonsingular at all points along C.

4.2.2 Rational Curves with Normals : Parametric Definition

When both a space curve and its associated "normal" are given in rational parametric form, their equations can be used directly to produce a linear system for interpolation, instead of first computing $(n+m-1)d+1$ points on the curve. Let $\mathbf{C} : (x = G_1(s), y = G_2(s), z = G_3(s))$ be a rational curve of degree d with associated "normals" $\mathbf{n}(s) = (n_x(s), n_y(s), n_z(s))$ of degree m. A Hermite interpolating surface $S : f(x, y, z) = 0$ of degree n which smoothly contains C is computed as follows:

1. **(containment condition)** Substitute $(x = G_1(s), y = G_2(s), z = G_3(s))$ into the equation $f(x, y, z) = 0$. This results in $nd+1$ homogenous linear equations as in section *4.1.2*.

2. **(tangency condition)**

   (a) Compute $\nabla f(s) = \nabla f(G_1(s), G_2(s), G_3(s))$ and $\mathbf{t}(s) = (\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds})$. Note $\mathbf{t} = (t_x, t_y, t_z)$ is the tangent vector to C.

   (b) i. If $t_x \neq 0$, use the equation $f_y(s) \cdot n_z(s) - n_y(s) \cdot f_z(s) = 0$.

   ii. If $t_y \neq 0$, use the equation $f_x(s) \cdot n_z(s) - n_x(s) \cdot f_z(s) = 0$.

   iii. If $t_z \neq 0$, use the equation $f_x(s) \cdot n_y(s) - n_x(s) \cdot f_y(s) = 0$.

   In each case, the numerator of the simplified rational function equation is set to zero. This yields ,additionally $(n-1)d+m+1$ linear equations in the coefficients of the surface $S : f(x, y, z) = 0$.

3. In total we obtain a homogeneous system of $(2n-1)d+m+2$ linear equations. Any non-trivial solution of the linear system, for which additionally ∇f is not identically zero for all points of C, (that is, the surface S is not singular along the curve C), will represent a surface which Hermite interpolates C.

The proof of correctness of the above algorithm follows from Lemma 4.2 and the following lemma, which shows why the selected equation of step *2.* satisfies the tangency condition.

Lemma 4.4 *If we choose a nontrivial solution for which the resulting Hermite interpolating surface S is nonsingular along the given curve C, then the step 2. guarantees that the tangency condition of definition 2.2 is met.*

Proof : The proof is similar to the proof of lemma 4.3 with minor modifications. We omit this here and refer the reader to the full paper. ♠

5 Applications: Mixed Points and Space Curve Data

The basic mechanics of Hermite interpolation using algebraic surfaces, as presented in the algorithms of sections *3.* and *4.*, are

1. properties of a surface to be designed are described in terms of a combination of points, curves, and possibly associated "normal" directions,

2. these properties are translated into a homogeneous linear system of equations with extra surface constraints, and then

3. nontrivial solutions of the above system are computed using the smallest surface degree

In particular the total number of linear equations generated for a possible algebraic surface of degree n to ***smoothly contain*** k points with fixed constant "normal" directions and also to ***smoothly contain*** l space curves of degree d with assigned "normal" directions, varying as a polynomial of degree m, is $3k+(2n+m-1)dl+2l$. This number reduces to $3k+(2n-1)dl+ml+2l$ when all the space curves and associated "normals" are defined parameterically.

For a given configuration of points, curves and "normals" data the above interpolation scheme, allows one to both upper and lower bound the degree of Hermite interpolated surfaces.

1. **Lower Bound** Let k be the rank of a homogenous system of linear equations, derived for the given geometric configuration. The rank tells us the exact number of independent constraints on the coefficients of our desired algebraic surface. Dependencies arise from spatial inter-relationships of the given points and curves. From the rank then we can conclude that there exists no algebraic surface of degree less than or equal to n_0 where n_0 is the largest n such that $K < k$ with $K = \binom{n+3}{3} - 1$.

2. **Upper Bound** Alternatively, the smallest n can be chosen such that $K \geq k$, where again K is the number of independent coefficient unknowns and k is the rank of the above linear system. The nontrivial real solutions of the linear system represents a $K-k$ parameter family of algebraic surfaces of degree n which interpolates the given geometric data. We then select suitable real surfaces from this family, which additionally satisfy our nonsingularity and irreducibility constraints[4]

[4]However some of these interpolating surfaces might still not be suitable for the design application they were intended to benefit. These problems arise when the given points or curves are smoothly interpolated, however lie on separate real components of the same nonsingular, irreducible algebraic surface. We consider this problem again in section 7.

We now enumerate some results which lower bound the degree of feasible Hermite interpolated surfaces.

1. Two skewed lines in space with constant-direction normals cannot be Hermite interpolated with real quadrics. The only real quadric which satisfies both containment and tangency conditions reduces into two planes.

2. Two lines in space with constant-direction normals can be Hermite interpolated with a real quadric if and only if the lines are parallel or intersect at a point, and the normals are not orthogonal to the plane containing them. The real quadric is a "cylinder" when the lines are parallel and a "cone" when the lines intersect.

3. The minimum degree of a real algebraic surface, which Hermite interpolates two lines in space, one with a constant direction normal, the other with a linearly varying normal is three.

4. Two lines with linearly varying normals can be Hermite interpolated by a quadric in only some special cases. In general, a surface of at least degree three is needed. When real quadric surface interpolation is possible, the real quadric is either a hyperboloid of one sheet (the two lines may be parallel, intersecting, or skewed) or a hyperbolic paraboloid (the two lines can only be intersecting or skewed).

We exhibit the method of generating tight upper bounds on the degree, by constructing the lowest degree Hermite interpolated surfaces for "blending" and "joining" primary surfaces of solid models as well as for "fleshing" curved wireframe models of physical objects.

Example 5.1 *A Hyperboloid Patch for Smoothing the Intersection of Two Cylindrical Surfaces*

The case of two circular cylinders is a common test case for "blending" algorithms. Various different ways have been given, (for e.g. see [5,11,15]) for computing a suitable surface which "smoothes" or "blends" the intersection of two equal radius cylinders, $S_1 : x^2 + y^2 - 1 = 0$ and $S_2 : x^2 + z^2 - 1 = 0$. We consider an ellipse C_1 on S_1 (it is the intersection with the plane $3x + y = 0$), defined parameterically, $C_1 : (\frac{2t}{1+t^2}, \frac{-6t}{1+t^2}, \frac{1-t^2}{1+t^2})$ with associated rational "normal" $\mathbf{n}_1(t) = (\frac{4t}{1+t^2}, 0, \frac{2-2t^2}{1+t^2})$, and the ellipse C_2 on S_2 defined implicitly, $C_2 : ((y^2 + z^2 - 1 = 0, x + 3y = 0)$ with associated "normal" $\mathbf{n}_2(x,y,z) = (0, 2y, 2z)$. Both C_1 and C_2's "normals" are respectively chosen in the same direction as the gradients of thier corresponding containing surfaces S_1 and S_2. This ensures that any Hermite interpolating surface for C_1 and C_2 will also meet S_1 and S_2 smoothly along these curves. As a possible Hermite interpolant we consider a degree two algebraic surface $S : f(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + iz + j = 0$. Applying the method of section *4.2.2*, to S and C_1 results in 8 equations, 5 from the containment condition and 3 from the tangency condition. (Note: 5 equations are supposed to be generated, but 2 of these turn out to be degenerate). For C_2, we use the method of section *4.2.1*, and first compute L_c = {(0, 0, 1), (-3, 1, 0), (3, -1, 0), (-2.4, 0.8, -0.6), (2.4,, -0.8, -0.6)} and L_t = {[(0, 0, 1), (0, 0, 2)], [(-3, 1, 0), (0, 2, 0)], [(3, -1, 0), (0, -2, 0)], [(-2.4,0.8,-0.6), (0,1.6,-1.2)], [(2.4,-0.8,-0.6), (0,-1.6,-1.2)]}. For these lists, we get 10 equations, 5 from the containment condition and another 5 from the tangency condition. Hence, overall the linear system consists of 9 independent unknowns and 18 equations. The rank of this system is 9, and hence we get the unique surface soution $f(x,y,z) = x^2 + y^2 - 8z^2 + 6xy + 8 = 0$. This real quadric satisfies both the nonsingularity and irreducibility constraints. It is a hyperboloid of one sheet and the lowest degree surface which "blends" the intersection of the two cylinders. See Figure 1. at the end of the paper. ♣

Example 5.2 *A Cubic Interpolation Surface for Smoothly Joining Two Cylindrical Surfaces*

Another example for an interpolated surfaces arises when we consider computing the lowest degree surface which can smoothly join two truncated circular cylinders $S_1 : x^2 + z^2 - 1 = 0$ for $y \geq 2$ and $S_2 : y^2 + z^2 - 1 = 0$ for $x \geq 2$. Traditionally, this join has been achieved by using a quarter section of a torus (a degree four algebraic surface). This example was considered in [15] where a similar solution was obtained by finding low degree surface members in appropriate product ideals of the section curves. Here, we illustrate the Hermite interpolation technique which also proves that degree three, is the lowest degree algebraic surface to satisfy the smooth-join requirement for this configuration. As before, we take a circle $C_1 : (\frac{2t}{1+t^2}, 2, \frac{1-t^2}{1+t^2})$ on S_1 with the associated rational "normal" $\mathbf{n}_1(t) : (\frac{4t}{1+t^2}, 0, \frac{2-2t^2}{1+t^2})$ and the circle $C_2 : (2, \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$ on S_2 with the associated rational "normal" $\mathbf{n}_1(t) : (0, \frac{4t}{1+t^2}, \frac{2-2t^2}{1+t^2})$. Again, both C_1 and C_2's "normals" are respectively chosen in the same direction as the gradients of their corresponding containing surfaces S_1 and S_2. This ensures that any Hermite interpolating surface for C_1 and C_2 will also meet S_1 and S_2 smoothly along these curves. A degree two algebraic surface does not suffice for Hermite interpolation, since the rank of the resulting linear system is greater than 9, the number of independent unknowns. Next as a possible Hermite interpolant consider a degree three algebraic surface with 19 independent unknown coefficients. Applying the Hermite interpolation method of section *4.2.2*, to the curves results in 24 equations (28 equations are supposed to be generated, but 4 of the 28 are degenerate.). The rank of this linear system is 19, and thus there is a unique cubic Hermite interpolating surface, which is $f(x,y,z) = x^3 + y^3 + x^2y + xy^2 + xz^2 + yz^2 - 4x^2 - 4y^2 - 4z^2 - 4xy + 3x + 3y + 4$. See Figure 2. at the end of the paper. ♣

Example 5.3 *A Family of Cubic Surfaces which Flesh a Saddle Wireframe*

Consider a wireframe of a solid model consisting of a circle, a parabola , and two lines. Using Hermite interpolation, we find a 2 parameter family of cubic surfaces which "fleshes" this wire frame. Again we can show that no degree two algebraic surface can contain all these curves simultaneously. Consider the circle $C_1 : (\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, -1)$, the parabola $C_2 : (t, -2t^2 + 2, 1)$, the line $C_3 : (1, 0, t)$, and the line $C_4 : (-1, 0, t)$. As a possible Hermite interpolant consider a degree three algebraic surface with 19 independent unknown coefficients. Applying the Hermite interpolation method of section *4.1.2*, we obtain a homogeneous linear system of 22 equations. The rank of this system is 17, so there is a 2 parameter family of cubic Hermite interpolating surfaces which is $f(x,y,z) = -bx^2z + \frac{a-b}{2}y^2z - \frac{a+b+4c}{4}yz^2 - ax^2 - \frac{a-b}{2}y^2 - \frac{a+b}{4}yz + cy + bz + a$. A suitable real solution surface from this family is obtained for $a = 3$, $b = 2$, and $c = 1$, yielding $f(x,y,z) = 12 + 4y - 2y^2 - 9yz^2 - 8x^2z + 2y^2z - 12x^2 - 5yz + 8z = 0$. See Figure 3. at the end of the paper. ♣

6 Extensions and Related Techniques: Meshing Quadric Surface Patches

Solving a linear system of equations plays a key role in Hermite interpolation. In what follows, we give another approach of algebraic surface design where a *nonlinear* system of equations needs to be solved.

In Hermite interpolation, the linear equations generated represent the constraints to be met by a single interpolating surface. The larger the number of independent containment and tangency constraints, the higher the degree of the resulting interpolating surface. The total number of constraints depends largely on the degrees of the given curves and their "normals".

Since the number of terms in an algebraic surface increases as the cube of its degree, computation with high degree algebraic surfaces gets expensive and error prone. Hence, for good reasons we are advised to keep the degrees of our "blending", "joining" and "fleshing" surfaces as low as possible. The problem considered in this section is to Hermite interpolate, conic curves in space with (not necessarily one), but a combination of quadric surface patches which themselves meet smoothly along thier intersection curves. Such "smooth" meshing has been largely addressed by [12,13] amongst others, using the Bézier representations of surfaces.

We first state a useful theorem from algebraic geometry, observed and used independently by numerous authors in various alternate forms

Lemma 6.1 *Let $S : f(x,y,z) = 0$ be an irreducible quadric surface, and $Q : q(x,y,z) = 0$ be a plane which intersects S in a conic C. Then, another quadric surface $S_1 : f_1(x,y,z)$ is tangent to S along C if and only if there exists nonzero constants α, β (possibly complex) such that $f_1 = \alpha f + \beta q^2$.*

Proof: The proof may be found for example, in [10,14]. ♠

Since we are interested in interpolation with real surfaces, we may restrict α and β to be real numbers.

A related theorem can be derived for the quadric surface interpolation of two conics in space.

Lemma 6.2 *Consider quadrics $S_1 : f_1 = 0$, $S_2 : f_2 = 0$ and planes $Q_1 : q_1 = 0$, $Q_2 : q_2 = 0$. Let $C_1 : (f_1 = 0, q_1 = 0)$ and $C_2 : (f_2 = 0, q_2 = 0)$ be two conics in space. Then C_1 and C_2 can be Hermite interpolated by a quadric surface S if and only if there exist nonzero constants α_1, α_2, β_1, and β_2 (possibly complex) such that $\alpha_1 f_1 + \beta_1 q_1^2 - \alpha_2 f_2 - \beta_2 q_2^2 = 0$.*

Proof: Trivial. (Just apply Lemma 6.1 twice.) ♠

This theorem is constructive, in that, it again yields a system of equations and a direct way of computing a Hermite interpolating quadric surface. Furthermore a solution to the above equations, linear in the α's and β's, exists if and only if such an interpolating quadric surface exists. Again, when real surfaces are favorable, we require α_1, α_2, β_1, and β_2 to be real numbers.

Example 6.1 *Suppose $C_1 : (x^2 + z^2 - 1 = 0, 3x + y = 0)$, and $C_2 : (y^2 + z^2 - 1 = 0, x + 3y = 0)$. We get the following equation from Lemma 6.2: $(\alpha_1 + 9\beta_1 - \beta_2)x^2 + (\beta_1 - \alpha_2 - 9\beta_2)y^2 + (\alpha_1 - \alpha_2)z^2 + (6\beta_1 - 6\beta_2)xy + (\alpha_1 - \alpha_2) = 0$. This implies $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\alpha_1 = -8\beta_1$. When $\alpha_1 = -8$ and $\beta_1 = 1$, the interpolating surface is $x^2 + y^2 - 8z^2 + 6xy + 8 = 0$.*

In the Lemma 6.2 and the example, the two conics on the given quadric surfaces, S_1 and S_2, were fixed. If we have freedom to choose different intersecting planes Q_1 and Q_2 then we may be able to find a family of quadric interpolating surfaces. In this case, the equations of planes Q_1 and Q_2 would have unknown coefficients and the use of Lemma 6.2 would result in a nonlinear system of equations, linear in terms of α_1, α_2, β_1 and β_2, and quadratic in terms of the unknowns of the plane's equations. Now, rather than trying to find a single quadric surface, we can also extend the above Lemma, to construct two or more quadrics which smoothly contain two given conics in space, and furthermore themselves intersect in a smooth fashion. The following Lemma, which is constructive tells us how to go about this.

Lemma 6.3 *Let $C_1 : (f_1 = 0, q_1 = 0)$ and $C_2 : (f_2 = 0, q_2 = 0)$ be two conics in space. These two curves can be smoothly contained by two "smoothly intersecting" quadrics $S_1 : g_1 = a_1 f_1 + b_1 q_1^2 = 0$ and quadrics $S_2 : g_2 = a_2 f_2 + b_2 q_2^2$ if and only if there exist nonzero constants a_1, a_2, b_1, b_2, α, β, and a plane $Q : q(x, y, z) = 0$ such that $a_1 f_1 + b_1 q_1^2 - \alpha(a_2 f_2 + b_2 q_2^2) - \beta q^2 = 0$.*

Proof: It follows from Bezout's theorem for surface intersection, that two quadrics always intersect smoothly in a plane curve (either an irreducible conic or straight lines). Let the intersection curve lie on the unknown plane Q, then just apply Lemma 6.1 three times. ♠

The final equation of the above Lemma results in a nonlinear (cubic) system of equations which is linear in terms of the unknowns a_1, a_2, b_1, b_2, α, and β, and quadratic in terms of the unknown coefficients of the plane $Q : q = 0$.

Example 6.2 *Let conic C_1 be given by $f_1 = x^2 + y^2 - z^2 + 4xy + 4x + 4y + 3 = 0$ (a hyperboloid of one sheet) and $q_1 = x + y + 1 = 0$. Simiarly, let conic C_2 be given by $f_2 = 19x^2 + 10y^2 - 9z^2 + 38xy - 114x - 114y + 180 = 0$ (a hyperbooid of one sheet), $q_2 = x + y - 3 = 0$, and let the unknown plane be $P : ax + by + cz + d = 0$. Then the equation for the system of smooth interpolating quadrics $a_1 f_1 + b_1 q_1^2 - \alpha(a_2 f_2 + b_2 q_2^2) = \beta(ax + by + cz + d)^2$ results in a nonlinear system of 10 equations: $-\beta c^2 + 9a_2\alpha - a_1 = 0$, $-2b\beta c = 0$, $-2a\beta c = 0$, $-2\beta cd = 0$, $-b^2\beta - \alpha b_2 + b_1 - 10a_2\alpha + a_1 = 0$, $-2ab\beta - 2\alpha b_2 + 2b_1 - 38a_2\alpha + 4a_1 = 0$, $-2b\beta d + 6\alpha b_2 + 2b_1 + 114a_2\alpha + 4a_1 = 0$, $-a^2\beta - \alpha b_2 + b_1 - 19a_2\alpha + a_1 = 0$, $-2a\beta d + 6\alpha b_2 + 2b_1 + 114a_2\alpha + 4a_1 = 0$, and $-\beta d^2 - 9\alpha b_2 + b_1 - 180a_2\alpha + 3a_1 = 0$. This nonlinear system has a nontrivial solution (in the sense that a_1, a_2, and α are nonzero) : $a_1 = -a^2\beta$, $b_1 = 2a^2\beta$, $a_2 = -\frac{a^2\beta}{9\alpha}$, $b_2 = \frac{19a^2\beta}{9\alpha}$, and $b = c = d = 0$.[5] Hence, the two conics C_1 and C_2 are smoothly contained by quadrics $g_1 = 0$ and $g_2 = 0$, respectively, and which in turn, smoothly intersect in a conic in the plane Q. The real quadric $g_1 = x^2 + y^2 + z^2 - 1 = 0$ is a sphere, while the other real quadric $g_2 = y^2 + z^2 - 1$ is a cylinder. Note that the above solution implies that there is only one pair of real quadric surfaces which smoothly contain the given conics. Also, for this case, it can be shown that neither a single quadric nor a single cubic surface can Hermite interpolate the two given conics. Geometrically then, the two hyperboloids of one sheet are smoothly joined by a sphere and a cylinder. See Figure 4. at the end of the paper.*

The above method of Lemma 6.3 can also be straightforwardly extended to finding a mesh of n quadric surfaces which smoothly contain two given conics in space. Necessarily the complexity of the nonlinear system of equations also goes up.

[5] This nonlinear system was solved with the aid of MACSYMA, on a Symbolics 3650

7 Conclusion

We have implemented the Hermite interpolation algorithms, as presented in sections *3.* and *4.* The program takes as input any collection of geometric data points, curves, with and without asscociated "normals". Both implicit and rational parametric representations of the space curves and "normals" are allowed. The system solves the linear system of equations with symbolic coefficient unknowns, using a variant of the Gaussian elimination algorithm and based on [7]. The rank computation is done implicitly during the solution step. The result, when nontrivial solutions exist, have some symbolic coefficients and represent a family of interpolation surfaces. These coefficients are instantiated, and desirable nonsingular and irreducible, real algebraic surfaces are sought. We are currently improving this implementation to include, a more user-friendly method of instantiating the interpolated solution, as well as a way of automatically incorporating the nonsingular and irreducibility constraints.

There is however one other desirable constraint which is not easily handled. Some of the irreducible and nonsingular, real interpolating surfaces might not be suitable for the design application they were initially intended to benefit. For example, one possible Hermite interpolated solution for two parallel ellipses is a hyperboloid of two sheets, with one sheet containing one ellipse and the other sheet containing the other ellipse. We are unable at this time to generate an algebraic constraint on the coefficients of the interpolating surfaces which would ensure that all given points and curves lie on the same continuous real surface component.

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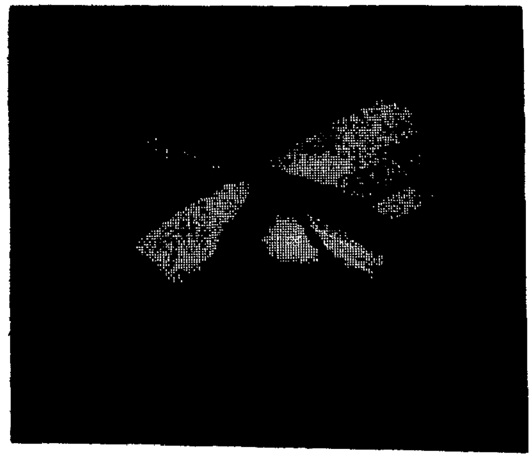

Figure 1.

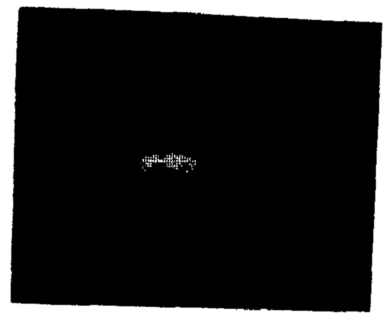

Figure 2.

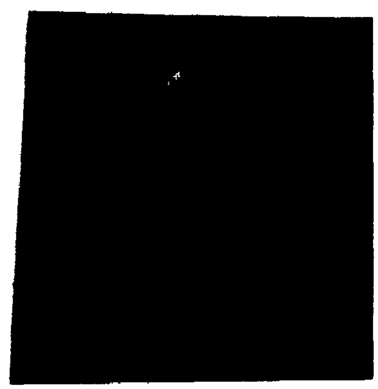

Figure 3.

Figure 4.