LOW DEGREE APPROXIMATION OF SURFACES FOR REVOLVED OBJECTS

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Abstract

We present a method for generating low degree C^k-continuous piecewise approximations of arbitrary algebraic surfaces of revolution. The approximating pieces are implicitly or parametrically defined algebraic surface patches. We show that degree d surface patches can be used for approximations with interpatch C^k continuity as high as $k = \lfloor \frac{(d+2)^2-12}{8} \rfloor$ for even d, and $k = \lfloor \frac{(d+1)(d+3)-12}{8} \rfloor$ for odd d. The method is based on a new technique to construct C^k-continuous implicit algebraic spline approximations of algebraic curves with the same degree and continuity tradeoff.

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1 Introduction

Algebraic curves and surfaces can be represented in an implicit form, and sometimes also in a parametric form. The implicit form of a real algebraic surface in $\mathbb{R}^3$ is

$$f(x, y, z) = 0 \tag{1}$$

where f is a polynomial with coefficients in $\mathbb{R}$. The parametric form, when it exists, for a real algebraic surface in $\mathbb{R}^3$ is

$$\begin{aligned} x &= \frac{f_1(s,t)}{f_4(s,t)} \\ y &= \frac{f_2(s,t)}{f_4(s,t)} \\ z &= \frac{f_3(s,t)}{f_4(s,t)} \end{aligned} \tag{2}$$

where the f_i are again polynomials with coefficients in $\mathbb{R}$. The *algebraic degree* of an algebraic curve or surface (in implicit or parametric form) is the *maximum* degree of any defining polynomial.

This paper presents two main ideas to be used in fitting low degree, piecewise algebraic surfaces (in the implicit or parametric form) to data sampled from arbitrary boundary surfaces of solids of revolution. One is the use of degree restricted bases for the piecewise approximation of the generating curve of revolution surfaces to yield approximating surfaces of the same algebraic degree as the degree of the piecewise curves. The other new idea arises in the development and use of C^k implicit algebraic splines for degree restricted interpolation and approximation of generating curves. The paper [8] studies a special family of implicit cubic curves which yields only tangent continuous cubic splines. While traditional fitting schemes are predominantly based on piecewise parametric representations[4, 7, 5, 9], we show here that implicit representations are also quite appropriate and in fact better equipped for restrictions on the bases and the degrees of the involved polynomials.

From Bezout's theorem[1], we realize that the intersection of two implicit surfaces of algebraic degree d can be a curve of geometric degree $O(d^2)$. Furthermore the same theorem implies that the intersection of two parametric surfaces of algebraic degree d can be a curve of degree $O(d^4)$. Hence, while the potential singularities of the space curve defined by the intersection of two implicit surfaces defined by polynomials of degree d can be as many as $O(d^4)$, the potential singularities of the space curve defined by the intersection of two parametric surfaces defined by polynomials of degree d can be as many as $O(d^8)$[2]. Hence keeping the degree of fitting surfaces as low as possible benefits both the efficiency and the robustness of post processing for modeling and display.

The rest of this paper is as follows. Section 2 characterizes the appropriate degree restricted bases for implicit and parametric algebraic curves which would yield revolution surfaces of the same algebraic degree as the degree of the curves. Section 3 characterizes C^k continuous piecewise surfaces of revolution and their construction from sampled data points. Section 4 describes the development and details for constructing cubic implicit algebraic C^1 and C^2 splines for approximating generating curves of surfaces of revolution.

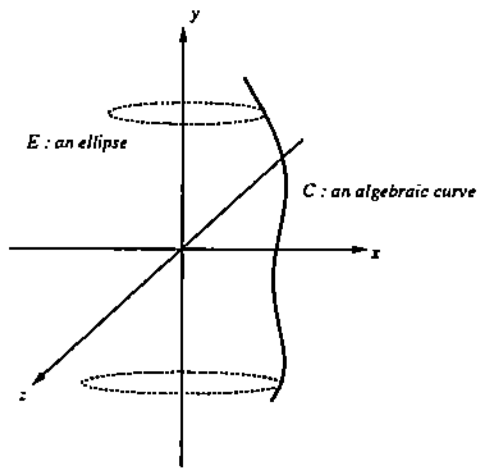


Figure 1: Revolution of an Algebraic Curve along an Ellipse

2 Surfaces of Revolution

2.1 Algebraic Surfaces of Revolution

Consider an algebraic surface which is obtained by revolving an algebraic curve $f(x,y) = 0$ (on the xy plane) around the y axis. (See Figure 1.) Rather than restricting ourselves to a circular rotation, we consider a more general elliptic revolution where the rotation path is described by an ellipse $E : x^2 + \frac{z^2}{\alpha^2} = \{r(y)\}^2$ with $\alpha > 0$. Here, $r(y)$ is the x coordinate of the point (x,y) on the curve $C : f(x,y) = 0$.

Now, the surface that results from revolving C along E is specified as "$x^2 + \frac{z^2}{\alpha^2} = \{r(y)\}^2$ *subject to* $f(r(y),y) = 0$." The equation $F(x,y,z) = 0$ of the surface S, hence, becomes $F(x,y,z) = f(\sqrt{x^2 + \frac{z^2}{\alpha^2}}, y) = 0$ where $F(x,y,z)$ is not necessarily *algebraic* due to introduction of the square root. By allowing only even-powered x's (x^0, x^2, x^4, $\cdots$) in $f(x,y)$, we can force $F(x,y,z)$ to be algebraic. Geometrically, this restriction, imposed on the revolved curve, that maintains *algebraicity*, means that the curve $f(x,y) = 0$ is symmetric to the y axis.

For quadric curves $f(x,y) = 0$, x^2 is the only possible factor of terms in f. Hence, f includes a 4-dimensional vector space V_f^2 of polynomials over real numbers that is spanned by the basis $\{x^2, y^2, y, 1\}$. In case of cubic curves $f(x,y) = 0$, the vector space V_f^3 is spanned by the basis $\{x^2y, x^2, y^3, y^2, y, 1\}$ with dimension 6. Quartic curves $f(x,y) = 0$ can be chosen from a more abundant vector space V_f^4 of dimension 9, generated by the basis $\{x^4, x^2y^2, x^2y, x^2, y^4, y^3, y^2, y, 1\}$. The bases of vector spaces V_f^d for higher degree curves are formulated in the same fashion.

Each algebraic curve of degree d in V_f^d, revolved around an ellipse, results in an algebraic surface of the same degree. Then we naturally come to the following question : "Is a surface, generated by revolving around an ellipse an algebraic curve that is not in V_f^d, algebraic at all?" In fact, the surface *is* algebraic, though the curve's degree gets doubled. This doubling of the degree arises from the single squaring required to remove the square root from odd-powered x factors. For example, consider a circle $f(x,y) = (x-5)^2+(y-5)^2-1 = x^2-10x+y^2-10y+49 = 0$

of radius 1, centered at $(5,5)$. This conic curve is not in V_f^2 because of the term $10x$. However, by moving $10x$ to the right hand side, and then squaring both sides, we can obtain a quartic curve in V_f^4 which generates a torus (of degree 4) by rotation. Intuitively, the squaring operation has an effect of putting another circle of the same shape to the other side of the y axis in order to artificially make the curve symmetric to the y axis. Any algebraic curve of degree d which is not in V_f^d can be made to be in V_f^{2d} by moving all terms with odd-powered x factors to one side, and squaring both sides.

REMARK 2.1. *Let $C : f(x,y) = 0$ be an algebraic curve of degree d, and $E : x^2 + \frac{z^2}{\alpha^2} = \{r(y)\}^2$ be an ellipse of a rotation path. Then, the algebraic surface $S : F(x,y,z) = 0$, resulting from revolving C around E, has degree d if C is symmetric around the y axis, or $2d$ otherwise.*

A geometric interpretation to Remark 2.1 is as follows : Consider a line on the xy plane parallel to the x axis. This line intersects with C at most d times. Now, imagine the intersection between the line and S. When C is symmetric, the number of intersection remains the same. However, if C is not symmetric, the number of intersection is doubled up because C, rotated by 180 degrees, creates the same number of line-curve intersections.

It is important to understand that, the degrees of freedom, in choosing a curve $f(x,y) = 0$ of degree d from V_f^d, is $\dim(V_f^d) - 1$ where $\dim(*)$ is the dimension of a vector space. Since all the polynomials on a line in V_f^d that passes through f and 0 describe the same curve, we have one less than $\dim(V_f^d)$ degrees of freedom. It is not hard to come up with the expression for $\dim(V_f^d)$:

$$\dim(V_f^d) = \begin{cases} \frac{(d+2)^2}{4} & \text{if } d \text{ is even} \\ \frac{(d+1)(d+3)}{4} & \text{if } d \text{ is odd} \end{cases}$$

In many situations as will be shown later, the curve $f(x,y) = 0$ is to be designed such that it satisfies given geometric requirements. We are interested in designing piecewise curves from given digitized data, and revolving them in a complicated manner to model some class of objects with low degree algebraic surfaces. It will be explained below how the degrees of freedom in piecewise algebraic curves of a given degree limit the geometric continuity between them.

EXAMPLE 2.1. Figure 2 (a) and (b) displays two quartic algebraic curves $(x^2+y^2)^2+3x^2y-y^3 = 0$ and $x^4 + x^2y^2 - 2x^2y - xy^2 + y^2 = 0$, respectively [12]. In Figure 3(a) and (b), shown are two surfaces revolved around $x^2 + z^2 = r(y)$. Their degrees are 4 and 8, respectively.

2.2 Parametric Surfaces of Revolution

Now, we get to a question : "Is it also possible to find a restricted bases of *rational parametric* curves that result in *rational parametric* surfaces of the *same geometric* degree after revolution around an axis?" Consider a rational parametric curve of degree d

$$C(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} \frac{x(t)}{w(t)} \\ \frac{y(t)}{w(t)} \end{pmatrix}$$

(a) (b)

Figure 2: Two Quartic Algebraic Curves

(a) (b)

Figure 3: A Degree 4 and a Degree 8 Algebraic Surface

where the degrees of the polynomials $x(t)$, $y(t)$, and $z(t)$ are at most d. The surface obtained by revolving $C(t)$ around y-axis along an ellipse $E : x^2 + \frac{y^2}{\alpha^2} = \{r(y)\}^2$ with $\alpha > 0$ can be represented as $F(s,t) = (X(s,t), Y(s,t), Z(s,t))$, where

$$
\begin{aligned}
X(s,t) &= \frac{2s}{1+s^2}\frac{x(t)}{w(t)} \\
Y(s,t) &= \frac{y(t)}{w(t)} \\
Z(s,t) &= \frac{\alpha(1-s^2)}{1+s^2}\frac{x(t)}{w(t)}.
\end{aligned}
$$

First, this representation answers that the revolved surface is always rational parametric. Then, the second question on the degree of $F(s,t)$ must be answered. We are interested in lowering both the *algebraic* degree in the polynomials in $F(s,t)$ and the *geometric* degree of $F(s,t)$ (the maximum possible intersection of $F(s,t)$ with a line). In construction of rational parametric revolved surfaces, we follow the same path we did in the previous subsection. From Remark 2.1, we know that an algebraic curve of degree d generates an algebraic surface of the same degree only when it is symmetric around an axis. Since every rational parametric curve of degree d is an algebraic curve of degree d, we are led to the fact that $F(s,t)$ is of degree d if $C(t)$ is symmetric around the y-axis.

A rational parametric curve is symmetric if there is a parametrization $C(t) = (X(t), Y(t)) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)})$ such that $X(t) = -X(-t)$ and $Y(t) = Y(-t)$. That is,

$$\frac{x(t)}{w(t)} = -\frac{x(-t)}{w(-t)} \tag{3}$$

$$\frac{y(t)}{w(t)} = \frac{y(-t)}{w(-t)} \tag{4}$$

The above conditions are met if *either* $x(t)$ is an odd function (all the terms with nonzero coefficients are odd-powered), and $y(t)$, $w(t)$ are even functions (all the terms with nonzero coefficients are even-powered), *or* $x(t)$ is an even function, and $y(t)$, $w(t)$ are odd functions. It is not difficult to see that the polynomials in the second case can be converted into the first case polynomials by multiplying t to both numerator and denominator, and vice versa. In fact, any polynomials that satisfies the conditions (3) and (4) fall in the above two categories.

LEMMA 2.1. *Let $x(t)$, $y(t)$, and $w(t)$ be polynomials in t such that $x(t)$ and $w(t)$ are relatively prime, and $y(t)$ and $w(t)$ are relatively prime. Then, $x(t)$ is an odd function, and $y(t)$, $w(t)$ are even functions if and only if $\frac{x(t)}{w(t)} = -\frac{x(-t)}{w(-t)}$ and $\frac{y(t)}{w(t)} = \frac{y(-t)}{w(-t)}$.*

PROOF : ($\Longrightarrow$) Trivial.
($\Longleftarrow$) Let $x(t) = x_e(t) + x_o(t)$ and $w(t) = w_e(t) + w_o(t)$, where $x_e(t)$, and $w_e(t)$ are even functions, and $x_o(t)$, and $w_o(t)$ are odd functions. From the first condition,

$$
\begin{aligned}
x(t)w(-t) + w(t)x(-t) &= x(t)(w_e(-t) + w_o(-t)) + (w_e(t) + w_o(t))x(-t) \\
&= x(t)(w_e(t) - w_o(t)) + (w_e(t) + w_o(t))x(-t)
\end{aligned}
$$

$$\begin{aligned} &= w_e(t)(x(t)+x(-t)) + w_o(t)(-x(t)+x(-t)) \\ &= 2(w_e(t)x_e(t) - w_o(t)x_o(t)) \\ &= 0. \end{aligned}$$

Hence, $w_e(t)x_e(t) - w_o(t)x_o(t) = 0$. Now, look closely at $w_e(t)$ and $x_e(t)$. First, both constant terms of $w_e(t)$ and $x_e(t)$ can not be nonzero at the same time. Or, the fact that $w_e(t)x_e(t)$ contains a nonzero term and $w_o(t)x_o(t)$ does not, leads to the contradiction because their difference can not be zero, as required by the above equations. Secondly, note that $X(0) = 0$. This requires that the constant term of $w_e(t)$ is nonzero, or $w(t)$ and $x(t)$ would have a common factor. Hence, $w_e(t)$ has a nonzero constant term and $x_e(t)$ does not.

Suppose that $w_o(t)$ is not a zero polynomial. Then, $x_o(t) = \frac{w_e(t)}{w_o(t)}x_e(t)$, and $x(t) = x_e(t) + x_o(t) = x_e(t)\frac{w_e(t)+w_o(t)}{w_o(t)} = w(t)\frac{x_e(t)}{w_o(t)}$. So, we are led to $x(t)w_o(t) = x_e(t)w(t)$. Existence of nonzero $x_e(t)$ and $w_o(t)$ contradicts to the fact that $x(t)$ and $w(t)$ are relatively prime because $w(t) \neq w_o(t)$. Hence, $w_o(t)$, and $x_e(t)$ are zero, implying that $w(t)$ is an even function, and $x(t)$ is an odd function. Now, from the second condition, $y(t)w(-t) - w(t)y(-t) = w_e(t)(y(t) - y(-t)) = 0$. Since $w_e(t) \neq 0$, $y(t) - y(-t) = 0$, hence $y(t)$ is an even function. □

From now on, we assume that $x(t)$ is an odd function, and $y(t)$ and $w(t)$ are even functions without loss of generality. Since a degree d curve $C(t) = (X(t), Y(t)) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)})$ is symmetric around y-axis, the surface made by revolving it around y-axis is a surface of geometric degree d. The surface equation $F(s,t)$ given above is represented by degree $d+2$ polynomials. In the below, we show it is possible to reduce the algebraic degree in the surface equation to d by applying a transformation to $F(s,t)$. Consider a new parametrization $F(u,v) = (\bar{X}(u,v), \bar{Y}(u,v), \bar{Z}(u,v)) = (\frac{\bar{x}(u,v)}{\bar{w}(u,v)}, \frac{\bar{y}(u,v)}{\bar{w}(u,v)}, \frac{\bar{z}(u,v)}{\bar{w}(u,v)})$. One transformation we use is $t = \sqrt{u^2+y^2}$. Geometrically, this transformation implies that only one half of a symmetric curve $C(t)$ is revolved. This removes the duplication caused by revolving the whole curve by 360 degrees, and possibly results in reduction of the algebraic degree in the surface equation. Note that $\bar{x}(u,v)$, $\bar{y}(u,v)$, $\bar{z}(u,v)$, and $\bar{w}(u,v)$ are all target polynomial in u, v we do not know yet. First, we require that $\bar{w}(u,v) = w(t) = w(\sqrt{u^2+v^2})$ which is algebraic because $w(t)$ is an even function. Secondly, we force that

$$\bar{Z}(u,v) = \frac{\alpha(1-s^2)}{1+s^2}\frac{x(t)}{w(t)} = \frac{\bar{z}(u,v)}{\bar{w}(u,v)}.$$

Solving it for s^2, $s^2 = \frac{\alpha x(t) - \bar{z}(u,v)}{\alpha x(t) + \bar{z}(u,v)}$. $\bar{z}(u,v)$ is still undetermined, and we have to choose an adequate polynomial for $\bar{z}(u,v)$. The minimum requirement for $\bar{z}(u,v)$ is that $\bar{X}(u,v) = \frac{2s}{1+s^2}\frac{x(t)}{w(t)}$ is rational. Let $x(t) = \sum_{i=0}^{l} x_{2i+1}t^{2i+1}$ where $2l+1 \leq d$. Now,

$$\begin{aligned} \frac{2s}{1+s^2}\frac{x(t)}{w(t)} &= \frac{\pm\sqrt{x(\sqrt{u^2+v^2})^2 - \bar{z}(u,v)^2}}{\alpha x(\sqrt{u^2+v^2})}\frac{x(\sqrt{u^2+v^2})}{\bar{w}(u,v)} \\ &= \pm\frac{\sqrt{x(\sqrt{u^2+v^2})^2 - \bar{z}(u,v)^2}}{\alpha\bar{w}(u,v)}. \end{aligned}$$

For the above expression to be rational, the expression inside the square root must be a perfect square. Since $x(\sqrt{u^2+v^2})^2 - \bar{z}(u,v)^2 = (u^2+v^2)(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)^2 - \bar{z}(u,v)^2$, choosing $\bar{z}(u,v) = v(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)$ ($\bar{z}(u,v) = u(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)$ is another possible symmetric choice.) results in a perfect square $u^2(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)^2$. Under this choice,

$$\begin{aligned}
\bar{X}(u,v) &= \frac{u(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)}{\alpha\bar{w}(u,v)} \\
\bar{Y}(u,v) &= \frac{y(\sqrt{u^2+v^2})}{\bar{w}(u,v)} \\
\bar{Z}(u,v) &= \frac{v(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)}{\bar{w}(u,v)}.
\end{aligned}$$

Also,

$$\begin{aligned}
s^2 &= \frac{\alpha x(\sqrt{u^2+v^2}) - v(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)}{\alpha x(\sqrt{u^2+v^2}) + v(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)} \\
&= \frac{\alpha(\sqrt{u^2+v^2} - v)(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)}{\alpha(\sqrt{u^2+v^2} + v)(\sum_{i=0}^{l} x_{2i+1}(u^2+v^2)^i)} \\
&= \frac{\sqrt{u^2+v^2} - v}{\sqrt{u^2+v^2} + v}.
\end{aligned}$$

REMARK 2.2. *Let $C : C(t) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)})$ be a rational parametric curve of degree d where $x(t)$ is an odd function, and $y(t)$, $w(t)$ are even functions, and $E : x^2 + \frac{z^2}{\alpha^2} = \{r(y)\}^2$ be an ellipse of a rotation path. Then, the algebraic surface $S : F(s,t) = (X(s,t), Y(s,t), Z(s,t))$ in the rational parametric form, resulting from revolving C around E, has the geometric degree d, and can be parameterized in the way that $X(s,t)$, $Y(s,t)$, and $Z(s,t)$ are degree d rational polynomials.*

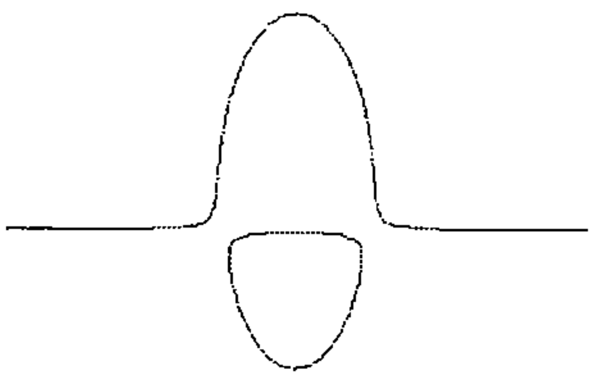
The class of the above rational parametric curves contains symmetric parametric curves that intersect with y-axis. The set of all such curves is only a proper subset of all symmetric parametric curves. Another interesting class of symmetric rational parametric curves is defined as $C(t) = (X(t), Y(t)) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)})$ such that $X(t) = -X(-\frac{1}{t})$ and $Y(t) = Y(-\frac{1}{t})$[1]. It still remains open how to specify all the bases of symmetric rational parametric curves of a given degree.

EXAMPLE 2.2. Recall the "three-leaf clover" in Example 2.1. Its parametric form is $C(t) = (\frac{t^3-3t}{t^4+2t^2+1}, \frac{t^4-3t^2}{t^4+2t^2+1})$. After circular revolution and the above reparametrization, the quartic surface is $F(u,v) = (\frac{u(u^2+v^2-3)}{(u^2+v^2)^2+2(u^2+v^2)+1}, \frac{(u^2+v^2)^2-3(u^2+v^2)}{(u^2+v^2)^2+2(u^2+v^2)+1}, \frac{v(u^2+v^2-3)}{(u^2+v^2)^2+2(u^2+v^2)+1})$.

3 Construction of Piecewise C^k Continuous Revolved Objects

So far we have discussed about revolution of a single algebraic curve, represented in either the implicit or the parametric form. A revolved object with a complicated shape, however, can

[1] For example, a hyperbola is in this class.



$f(x,y) = 0.481693x^2y + 2.7788x^2 + 0.0882735y^3 + 1.39039y^2 + 6.07771y + 5.65734 = 0$

Figure 4: A Nonparametric Algebraic Cubic Curve in V_f^3

not be modeled by rotating only one curve with a low degree. Instead, it is more appropriate to approximate a revolved object using surface slices meeting one by one with some order of geometric continuity. Hence, the revolved object design problem leads to a basic problem : design of piecewise C^k continuous curve segments.

In the below, we focus on design of piecewise C^k continuous *implicitly represented algebraic* curve segments.[2] Why algebraic? It is often stated that the class of rational parametric curves of a fixed degree is only a proper subset of the class of algebraic curves of the same degree. This implies that algebraic curves provide more flexibility in a design process. For example, while we use cubic curves in V_f^3 for C^1 objects, we observe that, in many cases, the curves are not singular, hence, nonparametric. (See Figure 4.) Also, a point can be easily classified as in, out, or on the boundary of an object that is made of several implicit algebraic curves and surfaces.

3.1 Algebraic Curves and Geometric Continuity

In this subsection, we describe how to compute two algebraic curves that meet with C^k continuity at a point. First of all, we assume the geometric information about a point p is expressed in terms of a (truncated) power series $C(t)$ of degree k, where $C(t) = (x(t), y(t)) = p + c_1t + c_2t^2 + \cdots + c_kt^k$, and $C(0) = p$. This truncated power series approximates the local geometric property (up to order k) of a curve about the point within a radius of convergence. (We will discuss later how this power series is computed.) Note that given an algebraic curve $f(x,y) = 0$ and a point $p = (p_x, p_y)$ on it, there is always a formal power series $C(t) = (x(t), y(t))$ about p such that $f(C(t)) \equiv 0$. In [3], a power series about a nonsingular point of an implicitly defined curve is obtained by repeatedly differentiating the implicit curve with its x and y substituted by a symbolic power series, and computing the power series' coefficients whose existence is guaranteed by the implicit function theorem. Newton's theorem, saying that every polynomial in y with coefficient polynomials or power series in x can be factored into linear power series factors in y, as can be seen as a generalization of the implicit function theorem, tells us how to find power series about both singular and nonsingular points on an algebraic curve [1].

[2] From now on, by "algebraic", we mean "implicit algebraic".

In our scheme, we go in the reverse direction : "Given a (truncated) formal power series $C(t)$ about a point p, find an algebraic curve $f(x,y)=0$ that is *faithful* to $C(t)$ at p." If the highest degree of terms in $C(t)$ is k, $f(x,y)=0$ is considered to meet $C(t)$ with C^k continuity at p. Let $f(x,y)=\sum_{i+j\leq d} a_{ij}x^i y^j = 0$ be an algebraic curve of degree d, and

$$C(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} p_x + c_{1x}t + c_{2x}t^2 + \cdots + c_{kx}t^k \\ p_y + c_{1y}t + c_{2y}t^2 + \cdots + c_{ky}t^k \end{pmatrix}$$

be a given parametric polynomial such that $C(0)=(p_x,p_y)\equiv p$. The relations on the coefficients of $f(x,y)$ can be extracted by repeatedly differentiating $f(C(t))$ up to order k, making all the derivatives vanish at $t=0$. The first few partial derivatives are :

$$\begin{aligned}
f(C(t))\,|_{t=0} &= f(p) = 0 \\
\frac{df(C(t))}{dt}\,|_{t=0} &= f_x(p)x'(0) + f_y(p)y'(0) \\
&= c_{1x}f_x(p) + c_{1y}f_y(p) = 0 \\
\frac{d^2f(C(t))}{dt^2}\,|_{t=0} &= f_{xx}(p)x'(0)^2 + 2f_{xy}x'(0)y'(0) + \\
&\quad f_{yy}(p)y'(0)^2 + f_x(p)x''(0) + f_y(p)y''(0) \\
&= c_{1x}^2 f_{xx}(p) + 2c_{1x}c_{1y}f_{xy}(p) + \\
&\quad c_{1y}^2 f_{yy}(p) + c_{2x}f_x(p) + c_{2y}f_y(p) = 0 \\
&\cdots
\end{aligned}$$

For each derivative of $f(C(t))$, a linear equation in terms of the unknown coefficients a_{ij} of f is generated, hence, any solution of the homogeneous linear system of $k+1$ equations becomes coefficients of algebraic curves of degree d meeting $C(t)$ with C^k continuity. Since an algebraic curve segment needs to satisfy the C^k conditions at both end points, $2k+2$ linear constraints must be satisfied. Hence, in order for an algebraic curve of degree d to exist, d must be chosen such that $\binom{d+2}{2} - 1 \geq 2k+2$, that is, the number of the degrees of freedom in coefficients of the curve is greater than or equal to the constraints for C^k continuity. Garrity and Warren [6] also discussed that the curve $f(x,y)=0$ and $C(t)$ meet with C^k continuity *if and only if* $f(C(t))$ and all of its derivatives up to order k vanish at $t=0$.

3.2 Computation of a Truncated Power Series

A truncated power series plays an essential role in computing piecewise C^k continuous algebraic curves. Hence, the question on how to get a truncated power series, in fact, a parametric curve of degree k, about a point, must be answered. One possible method is to generate a parametric curve interactively. For instance, a good user interface can be constructed where, say, a parabola for C^2 continuity is designed interactively and intuitively by using a mouse or typing in explicit values of the tangent and the curvature.

The finite difference method, as used when we make up our examples, is well suited when a curve to be rotated is given with regard to a sequence of digitized points. The digitized points near a point are a good source from which geometric nature can be extracted. There are various forms of divided-difference methods that extract geometric natures around a point from a given list of points [4]. In our case, we choose a parabola to locally approximate the points about a point, and take out tangential information from the parabola. Consider a sequence of points $\cdots, p_{i-2}, p_{i-1}, p_i, p_{i+1}, p_{i+2}, \cdots$ and an imaginary power series $C(t)$ from which, we assume, the digitized points near p_i come, and whose parameter value is $t = 0$ for p_i. Then, the tangent vector of $C(t)$ at $t = 0$ can be approximated by the approximation :

$$C'(0) \approx \frac{\sigma_i}{\mathsf{dist}(p_i, p_{i+1})}(p_{i+1} - p_i) + \frac{1-\sigma_i}{\mathsf{dist}(p_{i-1}, p_i)}(p_i - p_{i-1})$$

where $\sigma_i = \frac{\mathsf{dist}(p_{i-1}, p_i)}{\mathsf{dist}(p_i, p_{i+1}) + \mathsf{dist}(p_{i-1}, p_i)}$ and $\mathsf{dist}(*, *)$ is the distance between two points.

Repeatedly applying this approximation formula, we introduce a divided-difference :

$$\Delta^j p_l = \begin{cases} p_l & \text{if } j = 0 \\ \frac{1}{j}\left(\frac{\sigma_l}{\mathsf{dist}(p_l, p_{l+1})}(p_{l+1} - p_l) + \frac{1-\sigma_l}{\mathsf{dist}(p_{l-1}, p_l)}(p_l - p_{l-1})\right) & \text{if } j > 0 \end{cases}$$

Using this divide-difference operator, a truncated power series is represented as $C_i(t) = \Delta^0 p_i + \Delta^1 p_i t + \Delta^2 p_i t^2 + \cdots + \Delta^k p_i t^k$. Note that the geometric nature, stored in the coefficients of the power series is extracted from a sequence of $2k + 1$ neighboring points, centered at the junction point. This locality in the construction of a power series enables an interactive local modeling operation.

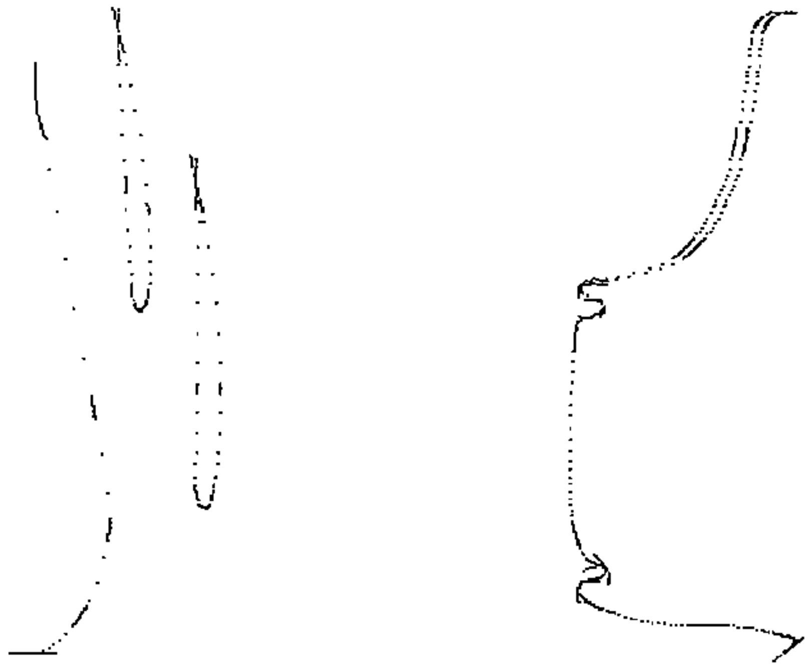
EXAMPLE 3.1. In Figure 5, two sets of digitized points are illustrated. (a) shows three lists of points that model engine parts[3], and (b) is a sequence of points that models a goblet. Each point sequence is displayed with truncated power series of order two at junction points.

3.3 Families of Algebraic Curves $f(x, y)$

In order to compute each curve segment $f_i(x, y) = 0$ that interplates two truncated power series $C_i(t)$ and $C_{i+1}(t)$ at two end points p_i and p_{i+1}, respectively, we construct a linear system $\mathbf{M_I x} = \mathbf{0}$ where the unknowns are coefficients of $f_i(x, y) = 0$. The linear system is made of $2(k + 1)$ equations that are generated for both truncated power series. Note that the rank of $\mathbf{M_I}$ must be less than the number of unknowns for a nontrivial solution to exist. Any nontrivial solution represents an algebraic curve that meets $C_i(t)$ and $C_{i+1}(t)$ at p_i and p_{i+1}, respectively, with C^k continuity.

In case all possible terms of degree d are used as a basis of $f_i(x, y) = 0$, then there are $\binom{d+2}{2}$ unknowns, and hence $\binom{d+2}{2} - 1$ degrees of freedom. For example, a cubic algebraic curve has ten unknown coefficients, hence, nine degrees of freedom. Since $8 (= 2(3 + 1))$ linear equations (some of them might be dependent on each other) needs to be satisfied for C^3 continuity, cubic piecewise algebraic curves can approximate a sequence of digitized points with C^3 continuity (at least, algebraically). However, if we choose a curve from V_f^d, we have fewer degrees of freedom due to restriction in the basis. There are only $\mathsf{dim}(V_f^d) - 1$ degrees of freedom for degree d, and this number must not be less than $2(k + 2)$, the maximum possible rank for a homogeneous linear system that needs to be satisfied for order k continuity. For instance, for C^1 continuity, at least, cubic curves are necessary, while order 2 continuity requires quartic curves.

[3] This data is originated from the 3D scanned engine data from NASA.



(a) (b)

Figure 5: Digitized Engine and Goblet with Truncated Power Series

Figure 6 (a) displays piecewise C^1 approximation with cubic algebraic curves in the restricted basis V_f^3. Note that a symmetric cubic curve in V_f^3 can have a tangent line parallel to x-axis only at the points on y-axis. Hence, the order of geometric continuity at the two junction points on the cowls around which the curve segments make vertical turnabouts. With symmetric quartic algebraic curves in V_f^4, it is possible to approximate the point data with C^2 continuity everywhere. (See Figure 6 (b).) For the goblet data, cubic curves in V_f^3, again, successfully model the data with C^1 continuity in Figure 7 (a). Figure 7 (b) shows a C^2 approximation of the same data with cubic curves in the general basis, which, hence, might not be symmetric around y-axis.

Now, when there are more degrees of freedom than the number of linear constraints, all the solutions in the null space of $\mathbf{M_I}$ algebraically interpolate two truncated power series with C^k continuity. However, it must be noted that every algebraic curve in the null space is not useful in the point of geometric modeling. A curve may not connect two end points, or could have a self-intersection along the segment. A heuristics to pick a nice curve segment is to generate a sequence of points between the end points that approximate a curve segment, and then, apply the least-squares approximation method to the points. In case of cubic algebraic curves, it is possible to state a condition on coefficients of cubic curves, in either the general or the restricted basis, that guarantees a smooth single curve segment inside a given control triangle as will be discussed in Section 4. In the example in Figure 7, control triangles are drawn together with curve segments where each curve segment was generated such that it subdivides its corresponding control triangle into a positive and a negative subspaces, and there exists only one smooth curve segment inside the control triangle. With help of the ability of subdivision

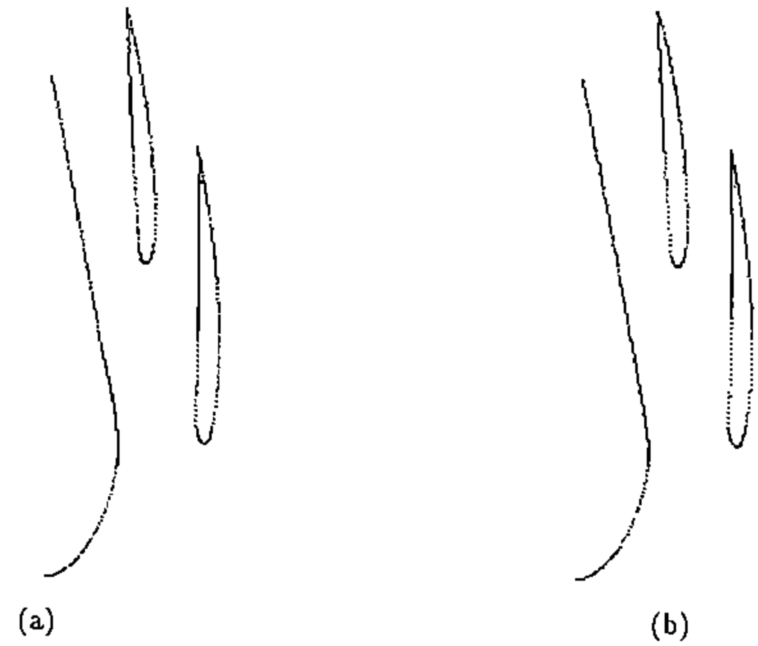


Figure 6: C^1 Cubic and C^2 Quartic Algebraic Curves

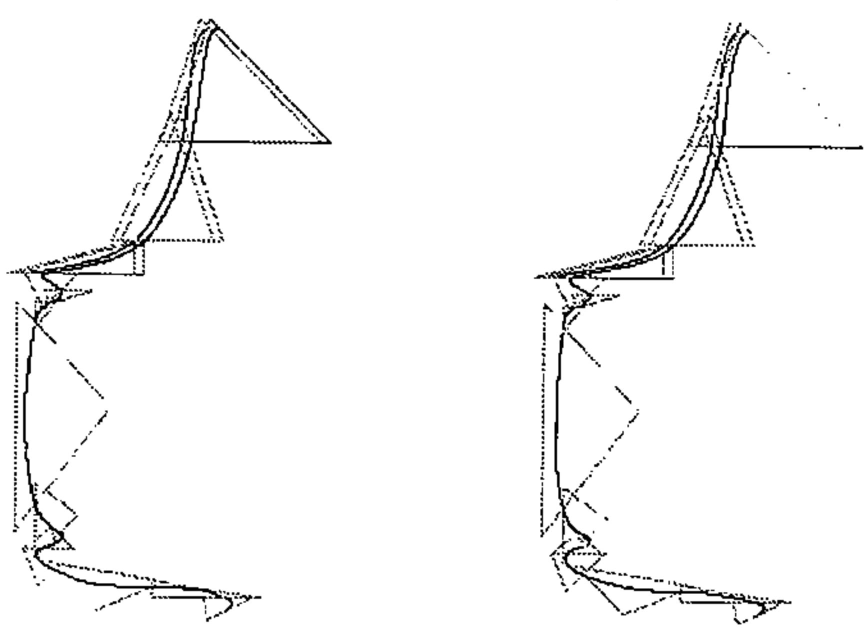
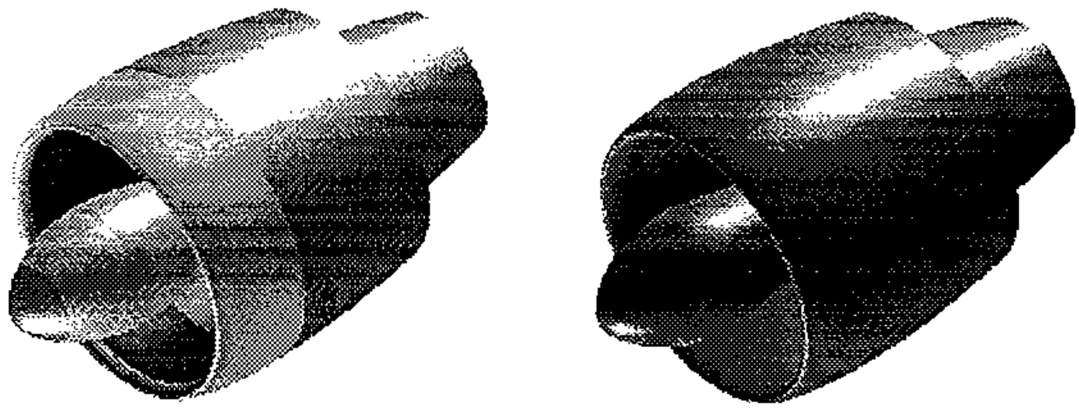


Figure 7: C^1 and C^2 Cubic Algebraic Curves



(a) (b)

Figure 8: C^1 Cubic and C^2 Quartic Algebraic Surface Models

of control triangles, the point classification operation for objects bounded by algebraic curves, and also, their rotated objects, is facilitated.

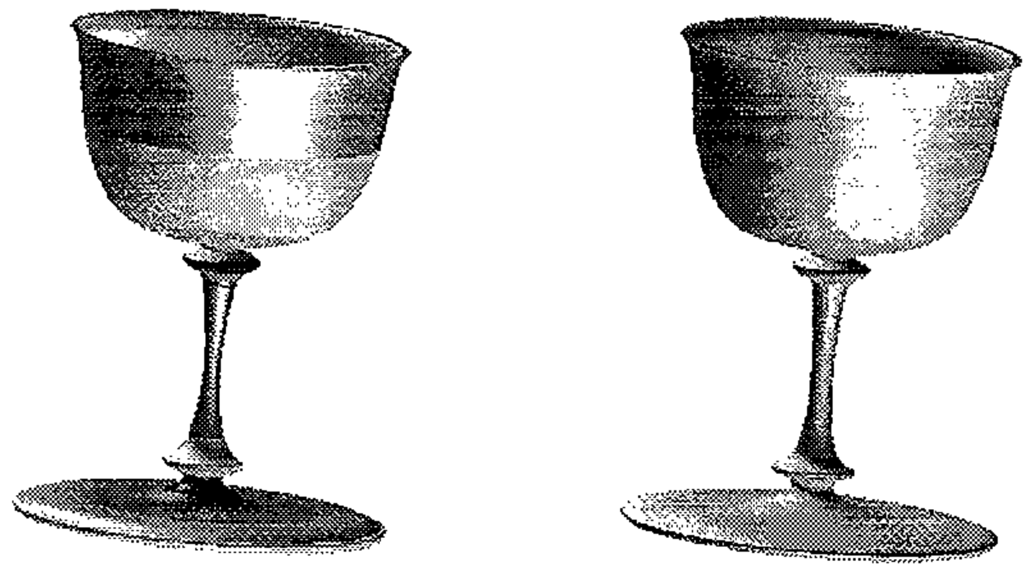
3.4 Piecewise C^k Continuous Revolved Objects

Once algebraic curve segments are computed, their revolved objects are easily obtained. C^1 approximation (except the two turnabout curves on the cowls) with cubic algebraic surfaces is shown in Figure 8 (a). Quartic algebraic surfaces approximate the same object well with C^2 continuity in Figure 8 (b). C^1 cubic algebraic goblet is illustrated in Figure 9(a). The C^2 goblet in Figure 9(b) is obtained by revolving the cubic curves in Figure 7 (b), and is made of degree 6 algebraic surfaces.

4 Cubic Algebraic Splines

In this section, we focus on implicitly defined cubic algebraic curves, and give conditions on the coefficients of cubic algebraic curves that guarantee nice properties inside regions bounded by triangles. These conditions are bases upon which robust C^1 cubic algebraic curves in the restricted basis and C^2 cubic algebraic curves in the general basis are constructed.

It must be noted that an algebraic spline that satisfies the *algebraic* constraints, as specified in Subsection 3.1, not necessarily possesses *geometrically* nice properties. It may be possible for an algebraic spline to have singular points between the end points or the spline may not connect the end points. Hence, extra efforts should be made to get an algebraic spine that is *effective* in the geometric modeling sense as well as to enforce continuity conditions. There are only a few works on cubic algebraic splines. Paluszny and Patterson [8] considered a special family of implicit cubic curves which yields only tangent continuous cubic splines. Our method differs in that tangents and curvatures are specified and controlled explicitly and algebraic splines are not limited to be convex inside bounding triangles.



(a) (b)

Figure 9: C^1 Cubic and C^2 Sextic Algebraic Surface Models

4.1 Algebraic Splines in Bernstein Basis

Barycentric coordinates in the plane are defined with respect to a nondegenerate triangle $\mathcal{T}$ having three vertices P_{00}, P_{n0}, P_{0n}. Any point P in the plane is uniquely expressed by the relation $P = uP_{n0} + vP_{0n} + (1-u-v)P_{00}$, where (u, v) is called the barycentric coordinate of P. The triangle vertices P_{00}, P_{n0}, and P_{0n} have barycentric coordinates $(0,0)$, $(1,0)$, and $(0,1)$, respectively. For more introduction to barycentric coordinates, see [5]. Given a triangle $\mathcal{T}$, a bivariate polynomial can be expressed using the Bernstein basis : $B^d(u,v) = \sum_{i+j\leq d} w_{ij} B^d_{ij}(u,v)$, where $B^d_{ij}(u,v) = \binom{d}{i,j} u^i v^j (1-u-v)^{d-i-j}$.

Sederberg [10] proposed to view an algebraic curve as the intersection of the explicit surface $w = B^d(u,v)$ with the plane $w = 0$, hoping to associate geometric meanings to the coefficients of the polynomial. Especially, the coefficients in the polynomial are considered as the w coordinates of the control net of a triangular Bernstein-Bézier surface patch, where the coefficient w_{ij} corresponds to the control point $b_{ij} = (\frac{i}{n}, \frac{j}{n})$ in the Bernstein basis. There is an one-to-one affine map between points in the power basis and in the Bernstein basis. Given the three vertices $P_{00} = \begin{pmatrix} p_{00x} \\ p_{00y} \end{pmatrix}$, $P_{n0} = \begin{pmatrix} p_{n0x} \\ p_{n0y} \end{pmatrix}$, and $P_{0n} = \begin{pmatrix} p_{0nx} \\ p_{0ny} \end{pmatrix}$, the map is described by $\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} p_{00x} \\ p_{00y} \end{pmatrix}$, where $M = \begin{pmatrix} p_{n0x} - p_{00x} & p_{0nx} - p_{00x} \\ p_{n0y} - p_{00y} & p_{0ny} - p_{00y} \end{pmatrix}$. It can be easily shown that there is also a linear mapping between the coefficients of the two equivalent bivariate polynomials, one in the power basis, and the other in the Bernstein basis.

As discussed before, C^k continuity of $f(x,y) = 0$ at a point p is achieved by making all the derivatives of $f(C(t))$ up to order k are zero at $t = 0$. Since the affine mapping between the Euclidean coordinates and the barycentric coordinates is diffeomorphic, C^k continuity can be obtained by assuring that all the derivatives of $B^d(C_B(t))$ up to order vanish at $t = 0$ where $B^d(u,v) = 0$ and $C_B(t)$ are algebraic and parametric curves in the Bernstein basis corresponding to $f(x,y) = 0$ and $C(t)$, respectively. From now on, we assume algebraic curves are described in the Bernstein basis.

4.2 Interpolation with Cubic Algebraic Curves

A *general* [4] cubic algebraic curve in the Bernstein basis is defined as $B^3(u,v) = \sum_{i+j\le 3} w_{ij} B^3_{ij}(u,v) = 0$. The coefficients w_{ij} is with respect to selection of a control triangle $\mathcal{T} = (P_{00}, P_{30}, P_{03})$ in the power basis. There are ten coefficients, and since dividing the equation out by a nonzero number would not change the algebraic curve, we see that there are nine degrees of freedom. While a *restricted* cubic algebraic curve in the Bernstein basis has the same form, there are extra linear dependency between w_{ij}'s where there are five degrees of freedom left. Hence, three degrees of freedom are left after C^2 interpolation with general cubic algebraic curves, and one for C^1 interpolation with restricted cubics.[5] In this section, we describe our idea with regard to C^2 continuous general algebraic cubics. Computation of C^1 continuous restricted algebraic cubic curves can be done along the same line.

Let $C_{B_0}(t)$ and $C_{B_1}(t)$ be two truncated power series of degree two that describe geometric properties at two points π_0 and π_1, respectively. One of the most important goals we try to reach is to find a triangle within which a single connected smooth piece of a cubic algebraic curve is confined such that the curve piece subdivides the triangle into a positive and a negative space. (See Figure 4.2.) For the sake of preciseness, we give the following definition :

DEFINITION 4.1. Let $\mathcal{T}$ be a triangle made of three vertices P_{00}, P_{n0}, P_{0n}. Consider a smooth curve segment on $B^n(u,v) = 0$ whose two end points are on the two sides $\overline{P_{00}P_{n0}}$ and $\overline{P_{00}P_{0n}}$. The curve segment is called *an effective algebraic spline associated with the bounding triangle* $\mathcal{T}$ if the curve segment intersects exactly once a line segment connecting P_{00} and any point on the side $\overline{P_{n0}P_{0n}}$.

The restriction imposed in the definition of an effective spline lets broken curve segments, loops, unwanted extra pieces and extraneous wiggles removed from our consideration, and also forces a spline curve segment subdivide a bounding triangle into a positive and a negative space. The ability of finding an effective spline with a proper bounding triangle is essential in that it allows easy implementations of many geometric operations possible. A point can be easily classified as in, out, or on the boundary of an object that is made of several algebraic splines. This point-classification operation is a primitive operation to high level geometric operations. Also, an effective spline curve can be graphed more efficiently.

Now, we attempt to confine a spline curve segment that connects π_0 and π_1 within the triangle $\mathcal{T}$. C^2 interpolation of the power series with a cubic polynomial generates six constrains, leaving three degrees of freedom. After solving the homogeneous linear system with ten unknowns, and six linearly independent constraints, the ten coefficients can be expressed in terms of linear functions in four free parameters λ_0, λ_1, λ_2, and λ_3. We have to decide if some

[4] We use the adjectives *general* and *restricted* to distinguish cubic algebraic curves in the general and the restricted bases, respectively.

[5] Counting shows C^3 interpolation is possible with general cubics. However, we limit ourselves to C^2 to be able to choose C^2 algebraic curves from abundant families of cubics.

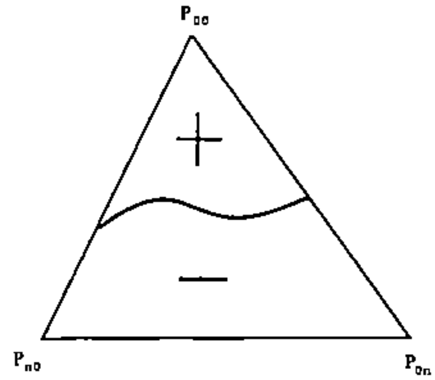


Figure 10: An Effective Spline Curve

appropriate values of λ_i can be found such that the intersection of the triangular Bernstein-Bézier patch defined by the computed coefficients with the $w = 0$ plane results in a single piece within $\mathcal{T}$. This is true if we are able to find some values for λ_0, λ_1, λ_2, and λ_3 such that the portion of the triangular patch corresponding to $\mathcal{T}$ cut through $\mathcal{T}$ *exactly once.* Definition 4.1 is translated into the following lemma :

LEMMA 4.1. *Let ten coefficients w_{ij} of $B^3(u,v)$ be expressed linearly in terms of λ_i, $i = 0,1,2,3$ after C^2 interpolation of $C_{B_0}(t)$ and $C_{B_1}(t)$ at π_0 and π_1, respectively, with respect to a control triangle $\mathcal{T}$. Then, there exists an effective cubic algebraic spline associated with $\mathcal{T}$ if and only if there exists some λ_i, $i = 0,1,2,3$ such that the univariate cubic polynomial $G(x) \stackrel{\text{def}}{=} B^3((1-\alpha)x, \alpha x) = g_3(\alpha)x^3 + g_2(\alpha)x^2 + g_1(\alpha)x + g_0(\alpha)$ has one and only one root in $0 \le x \le 1$ for all $\alpha \in [0,1]$.*

Proof : Define $L_\alpha(x) = ((1-\alpha)x, \alpha x)$, $0 \le x \le 1$ for some $0 \le \alpha \le 1$. L_α is the line segment connecting two points $(0,0)$ and $(1-\alpha, \alpha)$. Now, the intersection of $L_\alpha(x)$ with a curve segment on $B^3(u,v) = 0$ can be found by solving the cubic equation $B^3((1-\alpha)x, \alpha x) = 0$ in x. Hence, the curve segment intersects $L_\alpha(x)$ exactly once if and only if $B^3((1-\alpha)x, \alpha x) = 0$ has one and only one root in $0 \le x \le 1$. This proves the lemma. □

4.3 Negativity and Nonpositivity Conditions of a Polynomial

We briefly discuss mathematics on the negativity and nonpositivity conditions on the coefficients of a univariate polynomial in the closed interval $[0,1]$. This classical theorem [11] plays an important role in the proof of the forthcoming lemmas :

THEOREM 4.1. (DESCARTES' RULE OF SIGNS) *The number of positive real roots (multiplicities counted) of a polynomial with real coefficients, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, is never greater than the number of sign changes in the sequence of its coefficients $a_n, a_{n-1}, \cdots, a_1, a_0$, and, if less, then always by an even number.*

Descartes' rule of signs indicates an upper limit to the number of positive real roots while the number of sign variations is the exact number of positive real roots in case it is zero or one.

Note that an upper bound to the number of negative real roots of $f(x)$ can be obtained by replacing $f(x)$ by $f(-x)$, and zero is a real root just when $a_0 = 0$.

Now, we enumerate the lemmas on the negativity and nonpositivity of a univariate polynomial in the unit interval. These lemmas are used in computing all the values, if any, of λ_i, $i = 0, 1, 2, 3$, that gives effective splines with respect to $\mathcal{T}$. Their proofs are given in the full paper.

LEMMA 4.2. *A linear polynomial $f(x) = a_1x + a_0$ is negative for all x in the closed interval $[0, 1]$ if and only if ($a_0 < 0$)* and *($a_0 + a_1 < 0$).*

LEMMA 4.3. *A quadratic polynomial $f(x) = a_2x^2 + a_1x + a_0$ is negative for all x in the closed interval $[0, 1]$ if and only if either of the followings is true :*

- ($b_0 < 0$) and ($b_1 \leq 0$) and ($b_2 < 0$)
- ($b_0 < 0$) and ($b_1 > 0$) and ($b_2 < 0$) and ($4b_0b_2 - b_1^2 > 0$)

where $b_2 = a_0$, $b_1 = 2a_0 + a_1$, and $b_0 = a_0 + a_1 + a_2$.

LEMMA 4.4. *A cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is negative for all x in the closed interval $[0, 1]$ if and only if either of the followings is true :*

- ($b_0 < 0$) and ($b_1 \leq 0$) and ($b_2 \leq 0$) and ($b_3 < 0$)
- ($b_0 < 0$) and ($b_1 < 0$) and ($b_2 > 0$) and ($b_3 < 0$) and ($b_2^2 - 3b_3b_1 \leq 0$)
- ($b_0 < 0$) and ($b_3 < 0$) and ($b_1 > 0$ or $b_2 > 0$) and ($b_1 > 0$ or $b_2^2 - 3b_3b_1 > 0$) and ($-27b_0b_3^2 + 9b_1b_2b_3 - 2b_2^3 > 0$) and ($27b_0^2b_3^2 - 18b_0b_1b_2b_3 + 4b_1^3b_3 + 4b_0b_2^3 - b_1^2b_2^2 > 0$)

where $b_3 = a_0$, $b_2 = 3a_0 + a_1$, $b_1 = 3a_0 + 2a_1 + a_2$, and $b_0 = a_0 + a_1 + a_2 + a_3$.

Lemma 4.2, 4.3, and 4.4 for the negativity has the following companion lemmas for the nonpositivity whose proofs are omitted :

LEMMA 4.5. *A linear polynomial $f(x) = a_1x + a_0$ is nonpositive for all x in the closed interval $[0, 1]$ if and only if ($a_0 \leq 0$)* and *($a_0 + a_1 \leq 0$).*

LEMMA 4.6. *A quadratic polynomial $f(x) = a_2x^2 + a_1x + a_0$ is nonpositive for all x in the closed interval $[0, 1]$ if and only if either of the followings is true :*

- ($b_0 \leq 0$) and ($b_1 \leq 0$) and ($b_2 \leq 0$)
- ($b_0 < 0$) and ($b_1 > 0$) and ($b_2 < 0$) and ($4b_0b_2 - b_1^2 \geq 0$)
- ($b_0 < 0$) and ($b_1 > 0$) and ($b_2 = 0$) and ($b_1 + b_0 \leq 0$)

where $b_2 = a_0$, $b_1 = 2a_0 + a_1$, and $b_0 = a_0 + a_1 + a_2$.

LEMMA 4.7. *A cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is nonpositive for all x in the closed interval $[0, 1]$ if and only if either of the followings is true :*

- $(b_0 \leq 0)$ and $(b_1 \leq 0)$ and $(b_2 \leq 0)$ and $(b_3 \leq 0)$
- $(b_0 \leq 0)$ and $(b_1 < 0)$ and $(b_2 > 0)$ and $(b_3 < 0)$ and $(b_2^2 - 3b_3b_1 \leq 0)$
- $(b_0 \leq 0)$ and $(b_3 < 0)$ and $(b_1 > 0$ or $b_2 > 0)$ and $(b_1 > 0$ or $b_2^2 - 3b_3b_1 > 0)$ and $(-27b_0b_3^2 + 9b_1b_2b_3 - 2b_2^3 > 0)$ and $(27b_0^2b_3^2 - 18b_0b_1b_2b_3 + 4b_1^3b_3 + 4b_0b_2^3 - b_1^2b_2^2 \geq 0)$
- $(b_0 < 0)$ and $(b_1 > 0)$ and $(b_2 < 0)$ and $(b_3 = 0)$ and $(4b_0b_2 - b_1^2 \geq 0)$

where $b_3 = a_0$, $b_2 = 3a_0 + a_1$, $b_1 = 3a_0 + 2a_1 + a_2$, *and* $b_0 = a_0 + a_1 + a_2 + a_3$.

Note that by flipping the signs of coefficients of a polynomial, its positivity and nonnegativity conditions are easily derived from Lemma 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7.

4.4 Computation of Effective Cubic Algebraic Spline Curves

Back to Lemma 4.1, consider the univariate polynomial $G(x) = g_3(\alpha)x^3 + g_2(\alpha)x^2 + g_1(\alpha)x + g_0(\alpha)$. Substitution shows that $g_i(\alpha)$ is a polynomial of degree i in α involving the coefficients w_{ij}. Especially, $g_0(\alpha)$ is w_{00} which must not be zero or the line $L_0(t)$ in the proof of Lemma 4.1 would have two intersections with the Bernstein-Bézier patch. Hence, we can divide $B^3(u,v)$ by w_{00} without loss of generality. Geometrically, this means that we assign one to the weight, corresponding to P_{00}, of a control net of a Bernstein-Bézier patch, and algebraically, this means that we remove the redundancy among the ten coefficients of $B^3(u,v) = 0$.

The ten coefficients of $B^3(u,v)$ can be expressed linearly in terms of λ_0, λ_1, λ_2, and λ_3 by computing the four dimensional nullspace of a homogeneous linear system for C^2 interpolation. By transforming a basis of the nullspace, it is possible to have $w_{00} = \lambda_0$. Replacing λ_0 by one results in the coefficients w_{ij}, linearly expressed in λ_1, λ_2, and λ_3 only. Now,

$$\begin{aligned} H(x) &= h_3(\alpha)x^3 + h_2(\alpha)x^2 + h_1(\alpha)x + h_0(\alpha) \\ &\stackrel{\text{def}}{=} (x+1)^3 G(\frac{1}{x+1}) \\ &= g_0(\alpha)x^3 + (3g_0(\alpha) + g_1(\alpha))x^2 + (3g_0(\alpha) + 2g_1(\alpha) + g_2(\alpha))x \\ &\quad + g_0(\alpha) + g_1(\alpha) + g_2(\alpha) + g_3(\alpha) \end{aligned}$$

where $h_3(\alpha) = w_{00} = 1$, $h_2(\alpha) = (3w_{01} - 3w_{10})\alpha + 3w_{10}$, $h_1(\alpha) = (3w_{20} - 6w_{11} + 3w_{02})\alpha^2 + (6w_{11} - 6w_{20})\alpha + 3w_{20}$, and $h_0(\alpha) = (-w_{30} + 3w_{21} - 3w_{12} + w_{03})\alpha^3 + (3w_{30} - 6w_{21} + 3w_{12})\alpha^2 + (3w_{21} - 3w_{30})\alpha + w_{30}$.

First of all, $G(0) = g_0(\alpha) = h_3(\alpha) = 1 > 0$. Secondly, $G(1) = g_0(\alpha) + g_1(\alpha) + g_2(\alpha) + g_3(\alpha) = h_0(\alpha)$ must be negative in order for $G(x)$ to have exactly one root between zero and one. Third, the positive real roots of $H(x)$ are the real roots of $G(x)$ between zero and one. Hence, $G(x)$ has one and only one root in $[0,1]$ if and only if there exists exactly one positive real root of $H(x)$.

Consider the discriminant of the first derivative $H'(x) = 3h_3(\alpha)x^2 + 2h_2(\alpha)x + h_1(\alpha)$. Since $h_3(\alpha) > 0$, $H(x)$ has only one positive real root if it is nonpositive : $4h_2(\alpha)^2 - 12h_3(\alpha)h_1(\alpha) \leq 0$. In case $4h_2(\alpha)^2 - 12h_3(\alpha)h_1(\alpha) > 0$, $H(x)$ has the maximum value at $x_{max} = \frac{-h_2(\alpha) - \sqrt{h_2(\alpha)^2 - 3h_3(\alpha)h_1(\alpha)}}{3h_3(\alpha)}$. It is not hard to see that x_{max} is positive when and only when $h_2(\alpha) < 0$ and $h_1(\alpha) > 0$. Hence, when $h_2(\alpha) \geq 0$ or $h_1(\alpha) \leq 0$, $H(x)$ has one positive real root. (This case is when there is one sign change in the sequence of $H(x)$'s coefficients. By the Descartes' rule of sign, there exist exactly one positive real root.) When $h_2(\alpha) < 0$ and $h_1(\alpha) > 0$, we require that $H(x_{max}) < 0$. (This case is when there are three sign changes in the sequence of $H(x)$'s coefficients. By the Descartes' rule of sign, one or three roots are possible, and we make sure that there exists only one positive real root by this requirement.) Now,

$$H(x_{max}) = \frac{27h_0h_3^2 + 2(h_2^2 - 3h_1h_3)\sqrt{h_2^2 - 3h_1h_3} - 9h_1h_2h_3 + 2h_2^3}{27h_3^2} < 0,$$

hence, $2(h_2^2 - 3h_1h_3)\sqrt{h_2^2 - 3h_1h_3} < -27h_0h_3^2 + 9h_1h_2h_3 - 2h_2^3$. First of all, the right-hand side must be positive : $-27h_0h_3^2 + 9h_1h_2h_3 - 2h_2^3 > 0$. Since both sides are positive, they can be squared, and we get $27h_3^2(27h_0^2h_3^2 - 18h_0h_1h_2h_3 + 4h_1^3h_3 + 4h_0h_2^3 - h_1^2h_2^2) > 0$.

So, there are three cases :

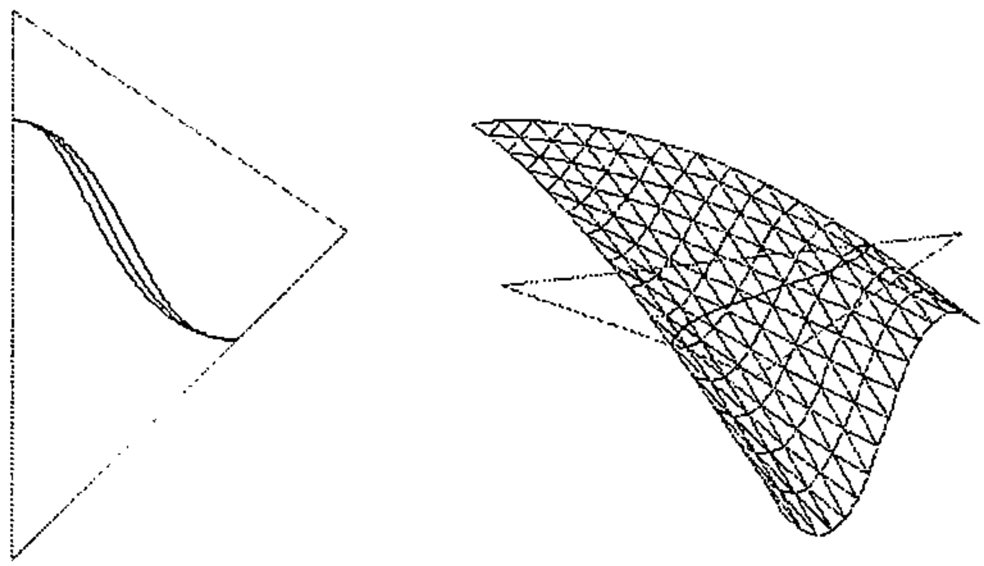
- [CASE 1] $h_3(\alpha) = 1 > 0$ and $h_2(\alpha)^2 - 3h_3(\alpha)h_1(\alpha) \leq 0$ and $h_0(\alpha) < 0$
- [CASE 2] $h_3(\alpha) = 1 > 0$ and $(h_2(\alpha) \geq 0$ or $h_1(\alpha) \leq 0)$ and $h_0(\alpha) < 0$
- [CASE 3] $h_3(\alpha) = 1 > 0$ and $h_2(\alpha) < 0$ and $h_1(\alpha) > 0$ and $h_0(\alpha) < 0$ and $h_2(\alpha)^2 - 3h_3(\alpha)h_1(\alpha) > 0$ and $(-27h_0(\alpha)h_3(\alpha)^2 + 9h_1(\alpha)h_2(\alpha)h_3(\alpha) - 2h_2(\alpha)^3) > 0$ and $(27h_0(\alpha)^2h_3(\alpha)^2 - 18h_0(\alpha)h_1(\alpha)h_2(\alpha)h_3(\alpha) + 4h_1(\alpha)^3h_3(\alpha) + 4h_0(\alpha)h_2(\alpha)^3 - h_1(\alpha)^2h_2(\alpha)^2) > 0$

Now, we are led to the following theorem :

THEOREM 4.2. *Let ten coefficients w_{ij} of $B^3(u,v)$ be expressed linearly in terms of λ_i, $i = 1,2,3$ with $w_{00} = 1$ after C^2 interpolation of $C_{B_0}(t)$ and $C_{B_1}(t)$ at π_0 and π_1, respectively, with respect to a control triangle $\mathcal{T}$. Then, there exists an effective cubic algebraic spline associated with $\mathcal{T}$ if and only if there exists some λ_i, $i = 1,2,3$ such that, for all $\alpha \in [0,1]$, either* [CASE 1], [CASE 2], *or* [CASE 3] *is satisfied.*

Note that Theorem 4.2 requires that *either* [CASE 1], [CASE2], *or* [CASE 3] is satisfied *for each* α in the interval $[0,1]$. For the sake of simple implementation, we use a bit stronger condition that *either* [CASE 1] is satisfied for all $\alpha \in [0,1]$ *or* [CASE 2] is satisfied for all $\alpha \in [0,1]$. $h_i(\alpha)$ is a polynomial in α of degree $3-i$ whose coefficients are linear functions of λ_1, λ_2, λ_3, and applying the lemmas in the previous subsection to [CASE 1] and [CASE 2] generates inequality constraints whose expressions are linear, quadratic, cubic, and quartic in λ_1, λ_2, λ_3. Hence, all the feasible solutions $(\lambda_1, \lambda_2, \lambda_3)$ of those constraints, if they exist at all, comprise a union of subspaces (could be null) in the three dimensional $\lambda_1\lambda_2\lambda_3$-space whose boundaries are linear, quadratic, cubic, or quartic algebraic surfaces. Choosing an effective cubic algebraic spline associated with a bounding triangle becomes equivalent to finding a feasible solution of the inequality constraints. Although this new condition find some subset of the whole subspace implied by the above corollary, our experiment lets us feel that the stronger condition is good enough to find effective algebraic splines.

EXAMPLE 4.1. In Figure 11(a), three instance cubic algebraic curves that C^2 interpolate two truncated power series $C_0(t) = (1+t, t^2)$ and $C_1(t) = (t, 1-2t^2)$ with respect to



(a) (b)

Figure 11: C^2 Continuous Cubic Algebraic Spline Curves

$\mathcal{T} = ((0.0, -1.0), (1.5, 0.5), (0.0, 1.5))$. The three curves chosen from the four dimensional space are $f_0(x, y) = 0.757333x^3 - 1.19933x^2y - 0.768667x^2 + 0.534667xy^2 + 0.2xy - 0.734667x + 0.004y^3 - 0.246y^2 - 0.504y + 0.746$, $f_1(x, y) = 4.08x^3 - 7.37x^2y - 5.99x^2 + 0.06xy^2 + 0.2xy - 0.26x - 1.42y^3 - 1.67y^2 + 0.92y + 2.17$, and $f_2(x, y) = 0.421333x^3 - 0.575333x^2y - 0.240667x^2 + 0.582667xy^2 + 0.2xy - 0.782667x + 0.148y^3 - 0.102y^2 - 0.648y + 0.602$. As C^2 continuity implies, $f_i(C_j(t)) = O(t^3)$, $i = 0, 1, 2$, $j = 0, 1$. Figure 11(b) illustrates how a cubic Bernstein surface patch is intersected with a bounding triangles to produce an effective cubic algebraic spline.

5 Conclusion

We have presented a comprehensive characterization of the appropriate degree restricted bases for implicit and parametric generating curves which would yield revolution surfaces of the same algebraic degree as the degree of the curves. Parametric spline curves with restricted bases can be constructed by adopting the well known techniques [4]. We presented details for constructing cubic implicit algebraic C^1 and C^2 spline curves. We are currently pursuing a natural generalization to higher degree implicit algebraic spline curves to achieve higher orders of continuity.

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