USING ALGEBRAIC GEOMETRY FOR MULTIVARIATE HERMITE INTERPOLATION

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Abstract

This paper uses some well known theorems of algebraic geometry to characterize polynomial Hermite interpolation in any dimension. Efficient numerical algorithms are presented for interpolatory curve fits through points in the plane, surface fits through points and curves in space, and in general, hypersuface fits through points, curves, surfaces, and sub-varieties in n dimensional space. These interpolatory fits may also be made to match derivative information at the data points.

*Dedicated to Professor Samuel Conte

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1 Introduction

Interpolation provides a direct way to fit analytic functions to sampled data. Motivated by computational efficiency, this paper deals with only polynomials as opposed to arbitrary analytic forms. One distinquishes between multivariate polynomial functions $\mathcal{F} : x_n = f_1(x_1, \ldots, x_{n-1})$, multivariate rational functions $\mathcal{R} : x_n = \frac{f_1(x_1,\ldots,x_{n-1})}{f_2(x_1,\ldots,x_{n-1})}$ and polynomial algebraic functions or implicitly defined hypersurfaces $\mathcal{H} : f_1(x_1, \ldots, x_n) = 0$, where all f_i are multivariate polynomials with coefficients in $\mathbb{R}$. While prior work on interpolation has dealt with multivariate polynomial functions $\mathcal{F}$ and rational functions $\mathcal{R}$, see for e.g. [1, 8, 10, 9], little work has been reported on interpolation with implicitly defined hypersurfaces $\mathcal{H}$. See [4] which summarizes prior work on implicit surface interpolation in three dimensions and provides several additional references.

This short paper presents a form of multivariate Hermite interpolation which generalizes the usual curve fits through points in the plane and surface fits through both points and curves in space to general hypersurface fits through points, curves, surfaces, and any sub-varieties upto dimension $n-2$ in n dimensional space together with the matching of specified derivative information along the sub-varieties. We show that even implicitly defined hypersurfaces $\mathcal{H}$ lend themselves quite naturally to Hermite interpolation in any dimension.

2 Preliminaries

In this section we review some basic definitions and theorems from algebraic geometry that we shall be using in subsequent sections. These and additional facts can be found for example in [12, 13].

The set of real and complex solutions (or *zero* set $Z(S)$) of a collection S of polynomial equations

$$
\begin{aligned}
\mathcal{H}_1 &: f_1(x_1, ..., x_n) = 0 \\
&\cdots \\
\mathcal{H}_m &: f_m(x_1, ..., x_n) = 0
\end{aligned}
\tag{1}
$$

with coefficients in $\mathbb{R}$ is referred to as an *algebraic set*. The algebraic set defined by a single equation ($m = 1$) is also known as a hypersurface. A algebraic set that cannot be represented as the union of two other distinct algebraic sets, neither containing the other, is said to be *irreducible*. An irreducible algebraic set is also known as an algebraic variety V.

A hypersurface in R_n, an n dimensional space, is of *dimension* $n-1$. The *dimension* of an algebraic variety V is k if its points can be put in $(1,1)$ rational correspondence with the points of an irreducible hypersurface in $k+1$ dimensional space. An algebraic set $Z(S)$ on the other hand may have irreducible components or *sub-varieties* of different dimension. An algebraic set is called *unmixed* if all of its sub-varieties are of the same dimension, and *mixed* otherwise. The *dimension* of the algebraic set $Z(S)$ is considered the maximum dimension of any of its sub-varieties. An algebraic variety of dimension 1 is also called an *algebraic space curve* and of dimension 2 is also called an *algebraic surface*. The following two lemmas summarize the resulting dimension of intersections of varieties and sub-spaces of different dimensions.

Lemma 2.1 *In R_n, an n dimensional space, a variety V_1 of dimension k intersects a general sub-space R_{n-k+h}, with $k > h$, in a variety V_2 of dimension h.*

Lemma 2.2 *In R_n, a variety V_1 of dimension k intersects a a variety V_2 of dimension h, with $h \geq n-k$, in an algebraic set $Z(S)$ of dimension at least $h+k-n$.*

In the above lemma, the resulting intersection is termed *proper* if all subvarieties of $Z(S)$ are of the same minimum dimension $h+k-n$. Otherwise the intersection is termed *excess* or *improper*.

The *degree* of an algebraic hypersurface is the maximum number of intersections between the hypersurface and a line, counting both real and complex intersections and at infinity. This degree is also the same as the degree of the defining polynomial. A degree 1 hypersurface is also called a *hyperplane*. The *degree* of an algebraic space curve is the maximum number of intersections between the curve and a hyperplane, counting both real and complex intersections and at infinity The *degree* of a variety V of dimension h in R_n is the maximum number of intersections between V and a sub-space R_{n-h}, counting both real and complex intersections and at infinity. The *degree* of an *unmixed* algebraic set is the sum of the degrees of all its sub-varieties.

The following theorem, perhaps the oldest in algebraic geometry, summarize the resulting degree of intersections of varieties of different degrees.

Theorem 2.1 (Bezout) *A variety of degree d which* properly *intersects a ~~g~~ variety of degree e does so either in an algebraic set of degree at most $d*e$ or infinitely often.*

The *normal* or *gradient* of a hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ is the vector $\nabla f = (f_{x_1}, f_{x_2}, \dots f_{x_n})$. A point $\mathbf{p} = (a_0, a_1, \dots a_n)$ on a hypersurface is a *regular* point if the gradient at $\mathbf{p}$ is not null; otherwise the point is *singular*. A singular point $\mathbf{q}$ is of multiplicity e for a hypersurface $\mathcal{H}$ of degree d if any line through $\mathbf{q}$ meets $\mathcal{H}$ in at most $d-e$ additional points. Similarly a singular point $\mathbf{q}$ is of multiplicity e for a variety V in R_n of dimension k and degree d if any sub-space R_{n-k} through $\mathbf{q}$ meets V in at most $d-e$ additional points. It is important to note that even if two varieties intersect in a *proper* manner, their intersection in general may consist of sub-varieties of various multiplicites. The total degree of the intersection, however is bounded by the above Bezout's theorem.

Finally, one notes that a hypersurface $f(x_1, ..., x_n) = 0$ of degree d has $\binom{n+d}{n}$ coefficients and one less than that number of independent coefficients.

3 Interpolation

Our first problem deals with constructing C^0 interpolatory hypersurfaces.

Problem 3.1 *Construct a single real algebraic hypersurface $\mathcal{H}$ in $\mathbb{R}^n$ which C^0 interpolates a collection of l_1 points $\mathbf{p}_i$, and l_k sub-varieties V_{j_k} of dimension $k-1$, $k = 2 \ldots n-1$ and degree $e[k]_{j_k}$.*

Since a point is a variety of dimension 0 and hypersurfaces in $\mathbb{R}^n$ are of dimension $n-1$, we note from Lemma 2.2 that a hypersurface in general will not intersect a given point. However, the hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d, can be made to contain i.e. C^0-interpolate the point $\mathbf{p_i}$ if the coefficients of f satisfy the linear equation $f(\mathbf{p_i}) = 0$.

Again from Lemma 2.2 we note that a hypersurface in $\mathbb{R}^n$ will always intersect all sub-varieties of dimension h, for $h = 1 \ldots n-2$ in algebraic sets of dimension at least $h-1$. To increase the dimension of the intersection or more precisely, to ensure that the hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d completely contain i.e. C^0-interpolates a sub-variety V of dimension h and degree $e[h]$ we do the following:

1. Select any set L_V of $d * e[h] + 1$ points on C, $L_V = \{\mathbf{p}_i = (x_i[i], \ldots x_n[i]) | i = 1, \ldots, d * e[h] + 1\}$. The set L_V may be computed by a straightforward generalization of computing points on algebraic curves and surfaces. See [3] for reference to such techniques.

2. Next, set up $d * e[h] + 1$ homogeneous linear equations $f(\mathbf{p}_j) = 0$, for $\mathbf{p}_j \epsilon L_V$. Any nontrivial solution of this linear system will represent an $\mathcal{H}$ which interpolates the entire subvariety V.

The proof of correctness of the above algorithm follows from Bezout's theorem 2.1. By making $\mathcal{H}$ contain $d * e[h] + 1$ points of V, ensures that $\mathcal{H}$ must intersect V infinitely often and since V is irreducible, $\mathcal{H}$ must contain the entire sub-variety.

The irreducibility of the sub-variety is not a restriction, since an algebraic set can be handled by treating each irreducible component separately. The situation is more complicated in the real setting, if we wish to achieve separate containment of one of possibly several connected real components of a single sub-variety. There is first of course the nontrivial problem of specifying a single isolated real component of the sub-variety. See [2] where a solution is derived in terms of a decomposition of space into cylindrical cells which separate out the various components of any real algebraic or semi-algebraic set.

For the collection of l_1 points $\mathbf{p}$, and l_k sub-varieties V_{j_k} of dimension $k-1$, $k = 2 \ldots n-1$ and degree $e[k]_{j_k}$ the above C^0 interpolation with a degree d hypersurface $\mathcal{H}$, yields a system M of $\sum_{k=1}^{n-1} l_k + \sum_{k=2}^{n-1} \sum_{j_k=1}^{l_k} d * e[k]_{j_k}$ linear equations. Remember $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d has $K = \binom{n+d}{n} - 1$ independent coefficient unknowns. C^0-interpolation of the entire collection of sub-varieties is achieved by selecting an algebraic hypersurface of the smallest degree n such that $K \geq r$, where r $(\leq k)$ is the rank of the system M of linear equations.

4 Hermite Interpolation

An algebraic hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ is said to Hermite interpolate or C^1-interpolate a sub-variety V with associated derivative or "normal" information $\mathbf{n}(\mathbf{p}) = (n_{x_1}(\mathbf{p}), \ldots n_{x_n}(\mathbf{p}))$, defined for points $\mathbf{p} = (x_1, \ldots x_n)$ on V if :

1. (containment condition) $f(\mathbf{p}) = 0$ for all points $\mathbf{p} = (x_1, \ldots x_n)$ of V.
2. (tangency condition) $\nabla f(\mathbf{p})$ is not identically zero and $\nabla f(\mathbf{p}) = \alpha \mathbf{n}(\mathbf{p})$, for some $\alpha \neq 0$ and for all points $\mathbf{p} = (x_1, \ldots x_n)$ of V.

Our second problem then deals with constructing C^1 interpolatory hypersurfaces.

Problem 4.1 *Construct a single real algebraic hypersurface $\mathcal{H}$ in $\mathbb{R}^n$ which C^1 interpolates a collection of l_1 points $\mathbf{p}_i$ with associated "normal" unit vectors $\mathbf{n}_i(\mathbf{p}_i)$, and l_k sub-varieties V_{jk} of dimension $k-1$ with $k = 2 \ldots n-1$ and degree $e[k]_{jk}$ together with associated "normal" unit vectors $\mathbf{n}[k]_{jk}$ for all points on each sub-variety of the given collection.*

In the previous section we have already shown that the containment condition reduces to solving a system of linear equations. We now prove that meeting the tangency condition for C^1-interpolation reduces to solving an additional set of linear equations.

A hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d, satisfies the tangency condition at the point $\mathbf{p}_i$ if the coefficients of f satisfy, without loss of generality, the $n-1$ homogeneous linear equations

$$n_{x_1} \cdot f_{x_i}(\mathbf{p}_i) - n_{x_i} \cdot f_{x_1}(\mathbf{p}_i) = 0 \quad i = 2 \ldots n$$

For the above equations we assumed, without loss of generality, that $n_{x_1} \neq 0$ as the given normal $\mathbf{n}$ is not identically zero at any point. To verify that the above equations correctly satisfy the tangency condition, it suffices to choose $\alpha = \frac{f_{x_1}}{n_{x_1}}$ for then each of the $f_{x_i} = \alpha n_{x_i}$. Also note that for the choice of $n_{x_1} \neq 0$, it must occur that $f_{x_1}(\mathbf{p}_i) \neq 0$, and hence $\alpha \neq 0$, for otherwise the entire $\nabla f(\mathbf{p})$ is identically zero.

To ensure that a hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d meets the tangency condition for C^1-interpolation of a sub-variety V of dimension h and degree $e[h]$ we do the following:

1. Select a set of L_{NV} of $(d-1) * e[h] + 1$ point-normal pairs $[\mathbf{p}_j, \mathbf{n}[h]_j]$ on V where $\mathbf{p}_i \epsilon L_V$, with point set L_V on V computed to meet the containment condition.
2. Substitute each point-normal pair in L_{NV} into the $n-h-1$ equations

$$n_{x_1} \cdot f_{x_i}(\mathbf{p}) - n_{x_i} \cdot f_{x_1}(\mathbf{p}) = 0 \quad i = 2 \ldots (n-h) \qquad (2)$$

to yield additionally $(n-h-1) * ((d-1) * e[h] + 1)$ linear equations in the coefficients of the $f(x, y, z)$.

The proof of correctness of the above algorithm follows from the following. We first note that even though each of the equations 2 above is evaluated at only $(d-1)*e[h]+1$ points of V it holds for all points on V. Each equation (2) defines an algebraic hypersurface T of degree $(d-1)$ which intersects V of degree $e[h]$ at, at most, $(d-1)e[h]$ points. Invoking Bezout's theorem, and from the irreducibility of V, it follows that V must lie entirely on the hypersurface T. Hence each equation (2) is satisfied along the entire sub-variety V.

We now show that the $n-h-1$ equations 2 satisfies the tangency condition as specified earlier. Again we assume, without loss of generality, that $n_{x_1} \neq 0$ as the given normal $\mathbf{n}$ is not identically zero at along V_h. Note that the containment i.e. C^0 interpolation of the dimension h variety V_h by the hypersurface $\mathcal{H}$ already guarantees that the h tangent directions on V_h at each point $\mathbf{p}$ of V_h are identical to h tangent directions of $\mathcal{H}$ at $\mathbf{p}$ on $\mathcal{H}$. Hence h components of the given normal vector $\mathbf{n}(\mathbf{p})$ (orthogonal to the tangent directions of V_h) are already matched with h components of the gradient vector $\nabla f(\mathbf{p})$ (orthogonal to the tangent directions of $\mathcal{H}$). Assume, without loss of generality, that these vector components are $f_{x_i} = \alpha n_{x_i}$, $i = (n-h+1)\dots n$, for any non-zero α. The remaining $n-h$ components of $\nabla f(\mathbf{p})$ of $\mathcal{H}$ are then matched up with the $n-h-1$ equations 2 as follows. Let $\alpha = \frac{f_{x_1}}{n_{x_1}}$. Then from the $n-h-1$ equations 2 we note that each of the $n-h-1$ $f_{x_i} = \alpha n_{x_i}$, $i = 2\dots(n-h)$ as required. Hence the entire vector $\nabla f(\mathbf{p}) = \alpha\mathbf{n}(\mathbf{p})$. Also note that for the choice of $n_{x_1} \neq 0$, it must occur that $f_{x_1}(\mathbf{p}_i) \neq 0$, and hence $\alpha \neq 0$, for otherwise the entire $\nabla f(\mathbf{p})$ is identically zero.

For the collection of l_1 points $\mathbf{p}$, and l_k sub-varieties V_{j_k} of dimension $k-1$, $k = 2\dots n-1$ and degree $e[k]_{j_k}$ to achieve the tangency condition with a degree d hypersurface $\mathcal{H}$, requires satisfying an additionaly system of $(n-1)*l_1 + \sum_{k=2}^{n-1}\sum_{j_k=1}^{l_k}(n-k-1)*((d-1)*e[k]_{j_k}+1)$ linear equations. For C^1 interpolation we obtain a single homogeneous system M of linear equations consisting of the linear equations for C^0 interpolation of section 3 together with the above linear quatins. Any non-trivial solution of this linear system M, for which additionally ∇f is not identically zero for all points of the collection, (that is, the hypersurface $\mathcal{H}$ is not singular at all points or along any of the subvarieties V_k), will represent a hypersurface which Hermite interpolates the collection.

5 Algorithmic Details

In this section, we discuss some computational aspects of Hermite interpolation, and give several examples of algebraic surface design with Hermite interpolation in three dimensional space. The basic method followed is:

1. properties of a surface to be designed are described in terms of a combination of points, curves, and possibly associated "normal" directions,
2. these properties are translated into a homogeneous linear system of equations with extra surface constraints, and then
3. nontrivial solutions of the above system are computed.

5.1 Computing Nontrivial Interpolation Solutions

As explained in the previous sections, Hermite interpolation algorithm converts geometric properties of a surface into a homogeneous linear system :

$$\mathbf{M}\mathbf{x} = \mathbf{0} \; (\mathbf{M} \in \mathbf{R}^{p\times q}, \mathbf{x} \in \mathbf{R}^q),$$

where p is the total number of equations generated, q is the number of unknown coefficients of a hypersurface of degree d in n dimensional space (hence, $q = \binom{n+d}{d}$), $\mathbf{M}$ is a matrix for linear equations, and $\mathbf{x}$ is a vector whose elements are unknown coefficients.

In order to solve the linear system in a computationally stable manner, we compute the singular value decomposition (SVD) of $\mathbf{M}$ [11]. Hence, $\mathbf{M}$ is decomposed as $\mathbf{M} = U\Sigma V^T$ where $U \in \mathbf{R}^{p\times p}$ and $V \in \mathbf{R}^{q\times q}$ are orthonormal matrices, and $\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_r) \in \mathbf{R}^{p\times q}$ is a diagonal matrix with diagonal elements $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$ ($r = min\{p, q\}$). It can be proved that the rank s of $\mathbf{M}$ is the number of the positive diagonal elements of Σ, and that the last $q - s$ columns of V span the null space of $\mathbf{M}$. Hence, the nontrivial solutions of the homogeneous linear system are compactly expressed as :

$$\{\mathbf{x}(\neq \mathbf{0}) \in \mathbf{R}^q \,|\, \mathbf{x} = \textstyle\sum_{i=1}^{q-s} r_i \cdot \mathbf{v}_{s+i}, \; \textit{where } r_i \in \mathbf{R}, \textit{ and } \mathbf{v}_j \textit{ is the jth column of } V\}.$$

Example 5.1 *Computing a Quadric Surface Interpolant*

Let $C : (\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, 0)$, and $\mathbf{n}(t) = (\frac{4t}{1+t^2}, \frac{2-2t^2}{1+t^2}, 0)$, which is from an intersection of a sphere $x^2 + y^2 + z^2 - 1 = 0$ with the plane $z = 0$. To find a surface of degree 2 which Hermite interpolates C, we let $f(x,y,z) = c_1x^2 + c_2y^2 + c_3z^2 + c_4xy + c_5yz + c_6zx + c_7x + c_8y + c_9z + c_{10}$. From the containment condition, we get 5 equations, $c_{10} - c_8 + c_2 = 0$, $2c_7 - 2c_4 = 0$, $2c_{10} - 2c_2 + 4c_1 = 0$, $2c_7 + 2c_4 = 0$, $c_{10} + c_8 + c_2 = 0$, and from the tangency condition, we also get 5 equations, $-2c_9 + 2c_5 = 0$, $-4c_6 = 0$, $-4c_5 = 0$, $4c_6 = 0$, $2c_9 + 2c_5 = 0$. The Σ in the SVD of $\mathbf{M}$ is $diag(5.65685, 4.89898, 4.89898, 2.82843, 2.82843, 2.82843, 2.0, 1.41421, 0.0, 0.0)$.[1] Hence, we see that the rank of $\mathbf{M}$ is 8, and the null space of $\mathbf{M}$ is $\mathbf{x} = r_1 \cdot \mathbf{v}_9 + r_2 \cdot \mathbf{v}_{10}$. The nontrivial solutions are obtained by making sure that the free parameters, r_1 and r_2, do not vanish simultaneously. Hence, the Hermite interpolating surface is $f(x,y,z) = 0.57735r_2x^2 + 0.57735r_2y^2 + r_1z^2 - 0.57735r_2 = 0$ which has one degree of freedom in controlling its coefficients. $f(x,y,z) = 0$ can be made to contain a point, say, $(1,0,1)$. That is, $f(1,0,1) = 0.57735r_2 + r_1 - 0.57735r_2 = r_1 = 0$. So, the circular cylinder $f(x,y,z) = 0.57735r_2(x^2 + y^2 - 1) = 0$ is an appropriate Hermite interpolating surface. □

5.2 Geometric Design Examples

These geometric design examples were generated using the special case of Hermite interpolation in three dimensional space. Further details of the implementations and additional examples are given in [5, 6].

Example 5.2 *A Quartic Surface for Blending Two Orthogonal Cylindrical Surfaces*

[1] The subroutine dsvdc of Linpack was used to compute the SVD of a matrix.

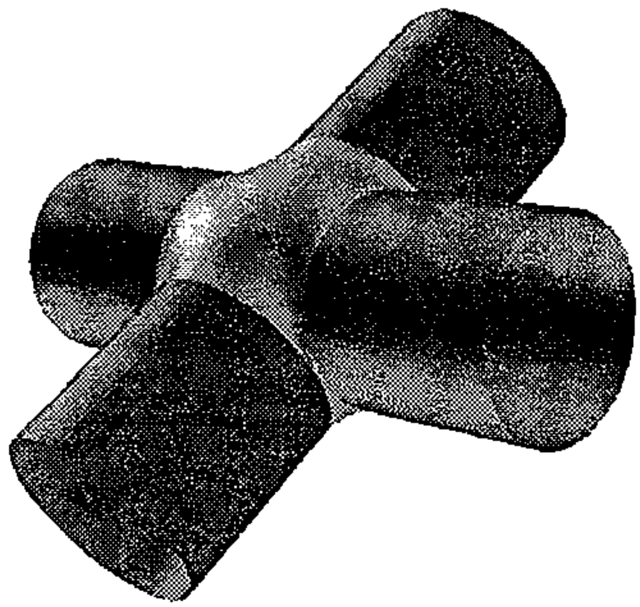

Figure 1: A C^1 Blend of Cylinders with a Quartic Surface

Here Hermite interpolation yields a quartic surface which smoothly blends two perpendicular cylinders. Input to Hermite interpolation is defined by $CYL_1 : x^2 + y^2 - 1 = 0$ for $z \geq 1$, $CYL_2 : x^2 + y^2 - 1 = 0$ for $z \leq -1$, $CYL_3 : y^2 + z^2 - 1 = 0$ for $x \geq 1$, and $CYL_4 : y^2 + z^2 - 1 = 0$ for $x \leq -1$.

Hermite interpolation produces 64 linear equations from the input, and the rank of $\mathbf{M_I} \in \mathbf{R}^{64\times 35}$ is 33. Hence, we find a 2-parameter (one degree of freedom) family of algebraic surfaces which is $f(x,y,z) = r_1 z^4 + (-r_2 - 2r_1)y^2z^2 + (-r_2 - 2r_1)x^2z^2 + r_2z^2 + (-r_2 - 3r_1)y^4 + (-r_2 - 2r_1)x^2y^2 + (2r_2 + 4r_1)y^2 + r_1x^4 + r_2x^2 - r_2$. An instance of this family ($r_1 = 10$, $r_2 = 1$) is shown in Figure 1.

Example 5.3 *A Quartic Interpolating Surface for a C^1 Join of Four Parallel Cylindrical Surfaces*

In this example, the lowest degree surface is constructed, which smoothly joins four truncated parallel circular cylinders defined by $CYL_1 : y^2 + z^2 - 1 = 0$ for $x \geq 2$, $CYL_2 : y^2 + z^2 - 1 = 0$ for $x \leq -2$, $CYL_3 : (y-4)^2 + z^2 - 1 = 0$ for $x \geq 2$, and $CYL_4 : (y-4)^2 + z^2 - 1 = 0$ for $x \leq -2$.

The C^1 interpolation technique shows that the minimum degree for such joining surface is 4, and finds a 2-parameter (one independent parameter) family of algebraic surfaces which is $f(x,y,z) = r_1(-0.316340556y^3 - 0.316340556yz^2 - 0.271753796x^2 + 0.553595973y^2 + 0.553595973z^2 + 0.316340556y - 0.049630952 + 0.033969224x^4 + 0.039542570y^4 + 0.079085139y^2z^2 +$

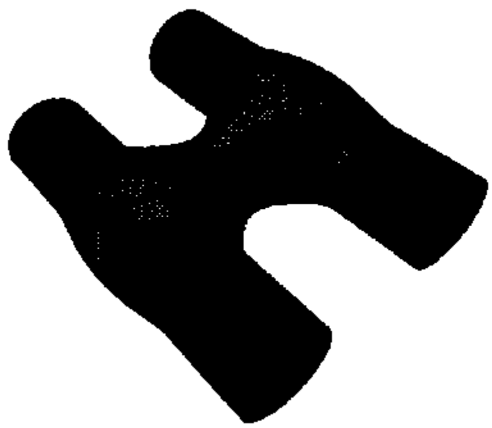

Figure 2: A C^1 Join of Cylinders with a Quartic Surface

$0.039542570z^4) + r_2(-0.025042574y^3 - 0.025042574yz^2 + 0.426400457x^2 + 0.043824505y^2 + 0.043824505z^2 + 0.025042574y - 0.899755741 - 0.053300057x^4 + 0.003130322y^4 + 0.006260644y^2z^2 + 0.003130322z^4)$

An instance of this family ($r_1 = 1$, $r_2 = 1.5$) is shown in Figure 3 and its use shown in Figure 2.

It should be noted that every instance is not always appropriate. The quartic surface in Figure 3 is the one used in Figure 2. One the other hand, the surface in Figure 4, which is not useful in light of geometric modeling, is also in the same family with $r_1 = 1$ and $r_2 = -1$.

Example 5.4 *Locally supported triangular C^1 interpolants for smoothing polyhedra*

The input is a convex polyhedron. First unique normals are chosen at the vertex endpoints, a necessary condition for obtaining a globally C^1 smooth polyhedra. Next a wireframe of conics are constructed where each conic replaces an edge and C^1 interpolates the corresponding vertices of the edge. Furtermore, normals are constructed for each curvilinear conic edge of the wireframe and varying quadratically along the conics. See Figure 5. Since the normals are quadratic functions and take on the value of the given normals at the vertex corners, specifying an additional normal vector at an interior point of each edge suffices.

The Hermite interpolation algorithm then constructs triangular C^1 interpolants - a 4 parameter family of quintic surfaces, one family per triangular facet of the wireframe. Instances of quintic surface patches generated for this example are displayed in Figure 6.

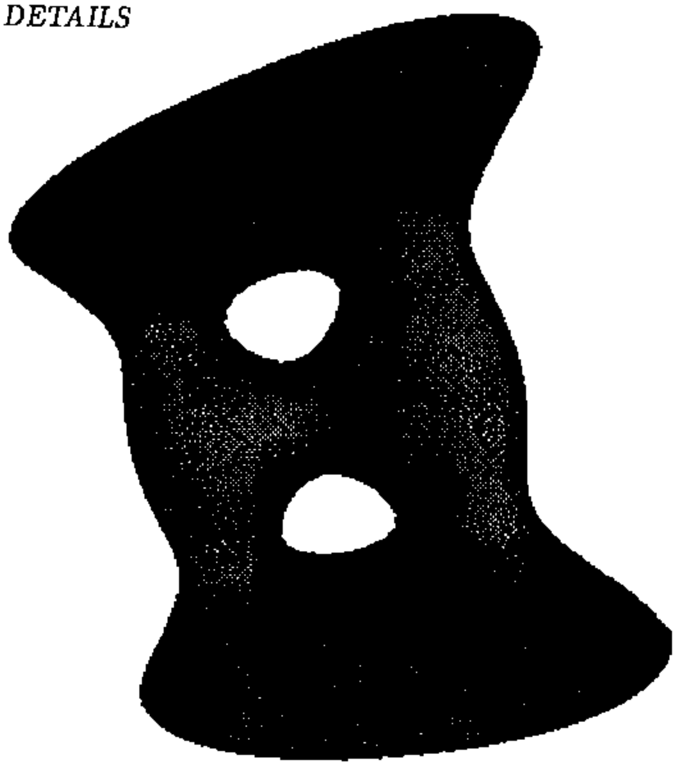

Figure 3: The C^1 Join Quartic Surface

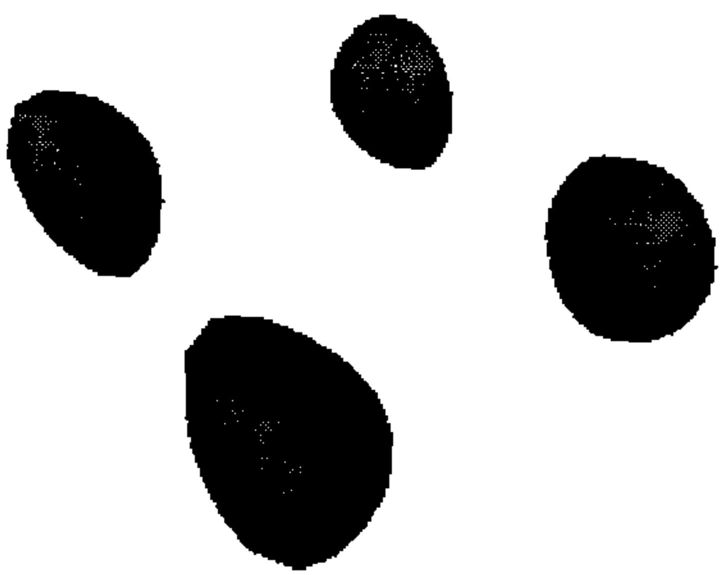

Figure 4: A Degenerate C^1 Join Quartic Surface

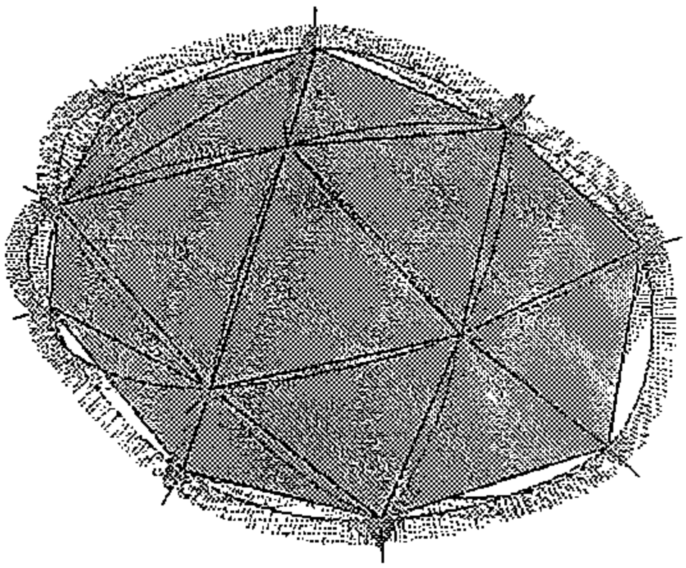

Figure 5: An input convex polyhedron with a C^1 conic wireframe

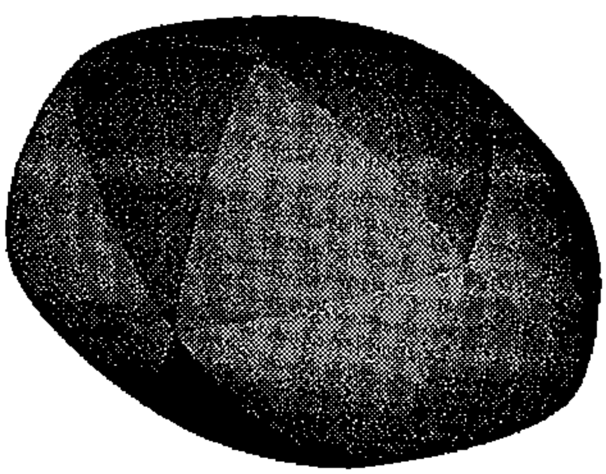

Figure 6: A smooth polyhedra with locally supported triangular C^1 interpolants

6 Conclusion

There are numerous open problems in the theory and application of multivariate interpolation. The primary problem amongst these stems from the non-uniqueness of interpolants in two and higher dimensions. There is an acute need for techniques of selecting a suitable candidate solution for the given input data, from the $K - r$ parameter family of C^1 interpolating hypersurfaces of degree d in n dimensional space. Here $K = \binom{n+d}{n} - 1$ and r is the rank of the system M of linear equations. One difficulty of the selection problem is exhibited in Figure 4 of example 5.3 of the previous section, where a certain choice of the free parameters of the interpolating surface family yields a degenerate joining solution in real space. Other difficulties arise from ensuring that the selected solution is also smooth (non-singular) in the domain of the input data.

One possible selection technique is the use of weighted least squares approximation on additional constructed data coupled with the interpolation of the given input data set [6, 7]. These and related problems are what we are currently pursuing in the study of multivariate Hermite interpolation.

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