Modeling with C^2 Quintic A-patches*

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Abstract

We present an efficient algorithm to construct C^2 smooth meshes of quintic A-patches that interpolate or approximate the vertices of a given triangulated polyhedron $\mathcal{P}$. An A-patch is a smooth and functional zero-contour patch of a trivariate polynomial in Bernstein-Bézier (BB) form defined within a tetrahedron.

A simplicial hull Σ is constructed based on $\mathcal{P}$. A quintic A-patch is then constructed within each tetrahedron of Σ so that the piecewise C^2 algebraic surface interpolates or approximates the vertices of $\mathcal{P}$ with given C^2 data. We guarantee that the algebraic surface is smooth and functional, namely, fully connected and free of singularity, unwanted branches.

1 Introduction

In this paper, we present an efficient algorithm to construct a C^2 smooth mesh with quintic A-patches to interpolate or approximate the vertices of a given polyhedron $\mathcal{P}$ with given C^2 data. The A-patch is a smooth and functional zero-contour patch of a trivariate polynomial in Bernstein-Bézier (BB) form defined within a tetrahedron[BCX95a], where "A" stands for algebraic. Solutions have been given to the problem of constructing a C^1 mesh of algebraic patches which *interpolate* the vertices of a *simplicial* polyhedron $\mathcal{P}$, by [Dah89] using quadric patches, [Guo91b, DTS93, Guo93, BCX95a] using cubic patches and [BI92b] using quintic for convex $\mathcal{P}$(all faces are triangular) and degree seven patches for arbitrary $\mathcal{P}$. While papers [Dah89, Guo91b, BI92b, DTS93, Guo93] provide heuristics based on monotonicity and least square approximation to circumvent the multiple sheeted and singularity problems of implicit patches, [BCX95a] introduces new sufficiency conditions for the BB form of trivariate polynomials within a tetrahedron, such that the zero contour of the polynomial is guaranteed functional and non-singular surface within the tetrahedron (the A-patch). The conditions are no more complicated than linear inequalities. [BX92] can be regarded as the two

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dimensional version of [BCX95a], where C^k continuous piecewise real algebraic curves(A-splines) are used to achieve local interpolation and approximation. In [BCX94, BCX95b], new schemes are given to construct an "inner" hull Σ of a polyhedron $\mathcal{P}$, such that the A-patch surface approximates $\mathcal{P}$ instead of interpolating its vertices.

In this paper, we present a scheme for building a C^2 patch complex with quintic surface patches. Like the C^1 scheme [BCX95a], a pair of tetrahedra are built on each side of each face of a triangulated polyhedron $\mathcal{P}$. We call them the face tetrahedra. A pair of tetrahedra are built to fill in the gap between each pair of face tetrahedra that share an edge and on the same side of $\mathcal{P}$. A quintic BB polynomial is defined within each tetrahedron so that they are C^2 continuous across their share boundaries. The zero contour of the piecewise C^2 polynomial C^2 interpolates the vertices of $\mathcal{P}$. We guarantee that the zero contour surface is non-singular and topologically equivalent to $\mathcal{P}$. If we use the schemes described in [BCX94, BCX95b] to construct the simplicial hull, then the final C^2 spline surface approximates $\mathcal{P}$ instead of interpolates it.

Related papers which approximate scattered data using algebraic patches are [Baj92, BBX94, BI92a, BIW93, MW91, Pra87, Sed90] and a classification of data fitting using parametric surface patches is given in [Pet90].

The rest of this paper is as follows. Section 2 gives some preliminary facts about Bernstein-Bezier (BB) representations, A-patches, the geometry of simple polyhedra and a definition of a simplicial hull. Section 3 presents details of the C^2 continuity schemes for quintic A-patches. Section 4 discuss a set of sufficient conditions which guarantee that the zero contour surface is functional. Section 5 provides some implementation details. And finally, Section 6 concludes the paper.

2 Notation and Preliminary Details

2.1 Bernstein-Bezier Representation and A-Patches

Let $\{p_1, \dots p_j\} \in I\!R^3$. Then the *convex hull* of these points is defined by $[p_1p_2...p_j] = \{p \in I\!R^3 : p = \sum_{i=1}^{j} \alpha_i p_i, \alpha_i \geq 0, \sum_{i=1}^{j} \alpha_i = 1\}$. Let $p_1,\ p_2,\ p_3,\ p_4 \in I\!R^3$ be affine independent. Then the tetrahedron(or three dimensional simplex) with vertices $p_1,\ p_2,\ p_3$, and p_4, is $V = [p_1p_2p_3p_4]$. For any $p = \sum_{i=1}^{4} \alpha_i p_i \in V$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is the barycentric coordinate of p. Let $p = (x, y, z)^T$, $p_i = (x_i, y_i, z_i)^T$. Then the barycentric coordinates relate to the Cartesian coordinates via the following relation

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \tag{2.1}$$

Any polynomial $f(p)$ of degree m can be expressed in Bernstein-Bezier(BB) form over V as $f(p) = \sum_{|\lambda|=m} b_\lambda\ B_\lambda^m(\alpha), \quad \lambda \in \mathcal{Z}_+^4$ where $B_\lambda^m(\alpha) = \frac{m!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!}\ \alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}\alpha_4^{\lambda_4}$ are the trivariate Bernstein polynomials for $|\lambda| = \sum_{i=1}^{4} \lambda_i$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$. Also $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is the barycentric coordinate of p, $b_\lambda = b_{\lambda_1\lambda_2\lambda_3\lambda_4}$(as a subscript, we simply write λ as $\lambda_1\lambda_2\lambda_3\lambda_4$) are called control points, and $\mathcal{Z}_+^4$ stands for the set of all four dimensional vectors with nonnegative

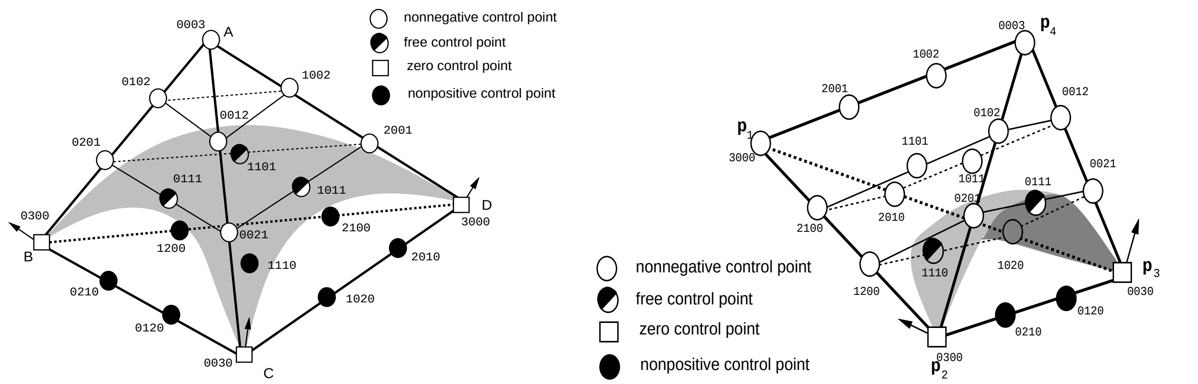


Figure 2.1: (a) A three sided cubic patch tangent at p_1, p_2, p_3 (b) A degenerate four sided cubic patch interpolates p_2 and p_3

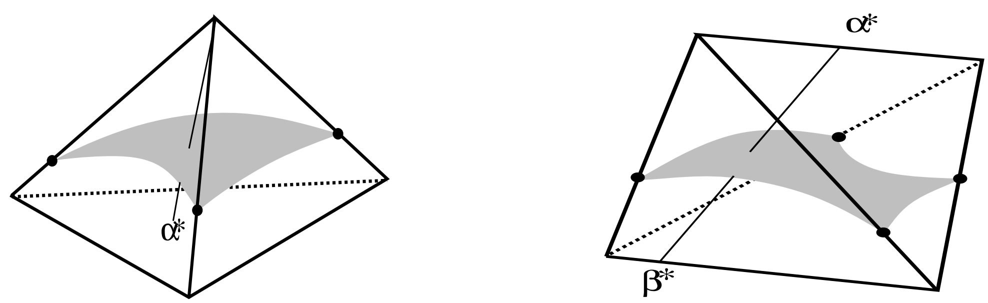


Figure 2.2: Three-sided and four-sided patches

integer components. Let

$$F(\alpha) = \sum_{|\lambda|=m} b_\lambda B_\lambda^m(\alpha), \ |\alpha| = 1, \tag{2.2}$$

be a given polynomial of degree m on the tetrahedron $S = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T \in I\!R^4 \ : \ \sum_{i=1}^4 \alpha_i = 1, \quad \alpha_i \geq 0\}$. The surface patch within the tetrahedron is defined by $S_f \subset S \ : \ F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$.

Definition 2.1 *Three-sided patch.*

Let the surface patch S_F be smooth on the boundary of the tetrahedron S. If any open line segment (e_j, α^*) with $\alpha^* \in S_j = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T : \ \alpha_j = 0, \alpha_i > 0, \ \sum_{i \neq j} \alpha_i = 1\}$ intersects S_F at most once(counting multiplicities), then we call S_F a three-sided j-patch (see Figure 2.2).

Definition 2.2 *Four-sided patch.*

Let the surface patch S_F be smooth on the boundary of the tetrahedron S. Let (i, j, k, ℓ) be a permutation of (1, 2, 3, 4). If any open line segment (α^*, β^*) with $\alpha^* \in (e_i e_j)$ and $\beta^* \in (e_k e_\ell)$ intersects S_F at most once(counting multiplicities), then we call S_F a four-sided ij-$k\ell$-patch (see Figure 2.2).

It is easy to see that if S_F is a four-sided ij-$k\ell$-patch, it is then also a ji-ℓk-patch, a ℓk-ji-patch, and so on. The Appendix contains proofs of the lemmas and theorems below.

Lemma 2.1 *The three-sided j-patch and the four-sided ij-$k\ell$-patch are smooth (non-singular).*

Proof. See [BCX95a]. $\diamond$

Theorem 2.1 *Let $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ satisfy the smooth vertex and smooth edge conditions and $j(1 \le j \le 4)$ be a given integer. If there exists an integer $k(0 \le k < n)$ such that*

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \ge 0, \qquad \lambda_j = 0, 1, \ldots, k-1, \tag{2.3}$$

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \le 0, \qquad \lambda_j = k+1, \ldots, n \tag{2.4}$$

and $\sum_{\substack{|\lambda|=n \\ \lambda_j=0}} b_\lambda > 0$ if $k > 0$, $\sum_{\substack{|\lambda|=n \\ \lambda_j=m}} b_\lambda < 0$ for at least one $m(k < m \le n)$, then S_F is a three-sided j-patch.

Theorem 2.2 *Let $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ satisfy the smooth vertex and smooth edge conditions and (i, j, k, ℓ) be a permutation of (1, 2, 3, 4). If there exists an integer $k(0 \le k < n)$ such that*

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \ge 0; \qquad \lambda_i + \lambda_j = 0, 1, \ldots, k-1, \tag{2.5}$$

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \le 0; \qquad \lambda_i + \lambda_j = k+1, \ldots, n \tag{2.6}$$

and $\sum_{\substack{|\lambda|=n \\ \lambda_i+\lambda_j=0}} b_\lambda > 0$ if $k > 0$, $\sum_{\substack{|\lambda|=n \\ \lambda_i+\lambda_j=m}} b_\lambda < 0$ for at least one $m(k < m \le n)$, then S_F is four-sided ij-$k\ell$-patch.

Proof. See [BCX95a]. $\diamond$

Lemma 2.2 *Let $f(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ be defined on the tetrahedron $[p_1p_2p_3p_4]$, then*

$$b_{(n-1)e_i+e_j} = b_{ne_i} + \frac{1}{n}(p_j - p_i)^T \nabla f(p_i), \quad j = 1, 2, 3, 4; \quad j \ne i \tag{2.7}$$

$$\begin{aligned} b_{(n-2)e_i+e_j+e_k} &= -b_{ne_i} + b_{(n-1)e_i+e_j} + b_{(n-1)e_i+e_k} \\ &+ \tfrac{1}{n(n-1)}(p_j - p_i)^T \nabla^2 f(p_i)(p_k - p_i), \quad j \ne i, k \ne i \end{aligned} \tag{2.8}$$

(2.7) can be found in [Guo91a](p.23). (2.8) is derived from directional derivative formulas (see [Far90] p.310).

Lemma 2.3 *([Far90] p.318) Let $f(p) = \sum_{|\lambda|=n} a_\lambda B_\lambda^n(\alpha)$ and $g(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ be two polynomials defined on two tetrahedra $[p_1p_2p_3p_4]$ and $[p_1'p_2p_3p_4]$, respectively. Then*
(i) f and g are C^0 continuous at the common face $[p_2p_3p_4]$ if and only if

$$a_\lambda = b_\lambda, \quad for \;\; any \;\; \lambda = 0\lambda_2\lambda_3\lambda_4, \quad |\lambda| = n \tag{2.9}$$

(ii) f and g are C^1 continuous at the common face $[p_2p_3p_4]$ if and only if (2.9) holds and

$$b_{1\lambda_2\lambda_3\lambda_4} = \beta_1 a_{1\lambda_2\lambda_3\lambda_4} + \beta_2 a_{0\lambda_2\lambda_3\lambda_4+0100} + \beta_3 a_{0\lambda_2\lambda_3\lambda_4+0010} + \beta_4 a_{0\lambda_2\lambda_3\lambda_4+0001} \tag{2.10}$$

(iii) f and g are C^2 continuous at the common face $[p_2p_3p_4]$ if and only if (2.9)-(2.10) holds and

$$\begin{aligned} b_{1\lambda_2\lambda_3\lambda_4} &= \beta_1^2 a_{2\lambda_2\lambda_3\lambda_4} + 2\beta_1\beta_2 a_{0\lambda_2\lambda_3\lambda_4+1100} + 2\beta_1\beta_3 a_{0\lambda_2\lambda_3\lambda_4+1010} + 2\beta_1\beta_4 a_{0\lambda_2\lambda_3\lambda_4+1001} \\ &+ \beta_2^2 a_{0\lambda_2\lambda_3\lambda_4+0200} + 2\beta_2\beta_3 a_{0\lambda_2\lambda_3\lambda_4+0110} + 2\beta_2\beta_4 a_{0\lambda_2\lambda_3\lambda_4+0101} \\ &+ \beta_3^2 a_{0\lambda_2\lambda_3\lambda_4+0020} + 2\beta_3\beta_4 a_{0\lambda_2\lambda_3\lambda_4+0011} + \beta_4^2 a_{0\lambda_2\lambda_3\lambda_4+0002} \end{aligned} \tag{2.11}$$

where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^T$ are defined by the following relation

$$p_1' = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + \beta_4 p_4, \quad |\beta| = 1$$

In Lemma 2.3, if $\beta_2 = \beta_3 = 0$, that is p_1', p_4 and p_1 are collinear, then (2.10) and (2.11) become

$$a_{0\lambda_2\lambda_3\lambda_4+0001} = \mu_1 a_{1\lambda_2\lambda_3\lambda_4} + \mu_2 b_{1\lambda_2\lambda_3\lambda_4} \tag{2.12}$$

$$\mu_1^2 a_{2\lambda_2\lambda_3\lambda_4} - \mu_1 a_{0\lambda_2\lambda_3\lambda_4+1001} = \mu_2^2 b_{2\lambda_2\lambda_3\lambda_4} - \mu_2 b_{0\lambda_2\lambda_3\lambda_4+1001} \tag{2.13}$$

respectively, where $\mu_1 = -\frac{\beta_1}{\beta_4}, \mu_2 = \frac{1}{\beta_4}$, that is $p_4 = \mu_1 p_1 + \mu_2 p_1'$.

2.2 Simplicial Hull

Definition 2.3 *Edge convexity*
Let $[p_ip_j]$ be an edge of a polyhedron $\mathcal{P}$ with endpoint vertex normals n_i and n_j. If $(p_j - p_i)^T n_i$ $(p_i - p_j)^T n_j \geq 0$, then the edge is convex. *Otherwise, it is* nonconvex. *If the edge satisfies the convex condition. and at least one of $(p_j - p_i)^T n_i$ and $(p_i - p_j)^T n_j$ is positive, then we say the edge $[p_ip_j]$ is* positively convex. *If both of them are zero then we say it is* zero convex. *If at least one of them is negative, the edge is* negatively convex.

Definition 2.4 *Face convexity*
Let $[p_ip_jp_k]$ be a triangular face of a polyhedron $\mathcal{P}$. If its three edges are non-negatively (positively or zero) convex and at least one of them is positive convex, then we say the face $[p_ip_jp_k]$ is positively convex. *If all the three edges are zero convex then the face is* zero convex. *If its three edges are non-positively (negatively or zero) convex and at least one of them is negatively convex, the face is* negatively convex. *Otherwise, $[p_ip_jp_k]$ is* non-convex.

Note, that here we are overloading the term *convex* to characterize the relations between the vertex normals and edges of faces. We distinguish between convex and non-convex faces in the simplicial hull below where we build one tetrahedra for convex faces and double tetrahedra for non-convex faces.

Definition 2.5 *Face tetrahedra*
A face-tetrahedron *$[p_ip_jp_kq_l]$ is a tetrahedron that is built based on a triangular face $[p_ip_jp_k] \in \mathcal{P}$. A face-tetrahedron $[p_ip_jp_kq_l]$ is* {positively | zero | negative | non-)} convex *if the face $[p_ip_jp_k]$ is {positively | zero | negative | non-)} convex.*

Definition 2.6 *Tangent containment*
A convex face-tetrahedron $[p_1p_2p_3p_4]$ is tangent-containing *if the tangent planes at the three interpolatory vertices p_1, p_2 and p_3 intersect with $[p_1p_2p_3p_4]$; A pair of non-convex face-tetrahedra $[p_1p_2p_3p_4q_4]$ is* tangent-containing *if the tangent planes at the three interpolatory vertices p_1, p_2 and p_3 intersect with either $[p_1p_2p_3p_4]$ or $[p_1p_2p_3q_4]$;*

Definition 2.7 *Simplicial hull*
A simplicial hull *of triangulation $\mathcal{T}$, denoted as $\Sigma = (\mathcal{S}_f, \mathcal{S}_e, \mathcal{R}_{tf}, \mathcal{R}_{fe})$ is defined as (1) $\mathcal{S}_f = \{[p_ip_jp_kq_l]\}$ is a collection of face tetrahedra. (2) $\mathcal{S}_e = \{[p_ip_jq_ks_l]\}$ is a collection of edge tetrahedra. (3) $\mathcal{R}_{tf} = \mathcal{T} \times \mathcal{S}_f$ is a relation between $\mathcal{T}$ and $\mathcal{S}_f$, which can be described as (i) (single sided) there is one* tangent-plane-containing *face-tetrahedron $[p_ip_jp_kq_l] \in \mathcal{S}_f$ is built on a convex face $[p_ip_jp_k] \in \mathcal{T}$ and (ii) (double sided) there are a* tangent-plane-containing *pair of nonconvex face-tetrahedra $[p_ip_jp_kq_l]$, $[p_ip_jp_k\overline{q_l}] \in \mathcal{S}_f$ are built on a nonconvex face $[p_1p_2p_3] \in \mathcal{T}$, one on each side;*
(4) $\mathcal{R}_{fe} = \mathcal{S}_f \times \mathcal{S}_f \times \mathcal{S}_e \times \mathcal{S}_e$ is a relation between a pair of neighboring face-tetrahedra and a pair of edge tetrahedra, which can be described as (i) (non-intersection) two face-tetrahedra $[p_ip_jp_kq_l]$, $[p_ip_jp_mq_n] \in \mathcal{S}_f$ that share a common edge $[p_ip_j] \in \mathcal{T}$ does not intersect each other and (ii) a pair of edge-tetrahedra $[p_ip_jq_ls_r]$, $[p_ip_jq_ns_r] \in \mathcal{S}_e$ where $s_r = \alpha q_l + (1-\alpha)q_n), 0 < \alpha < 1$, are built between the pair of edge-sharing face-tetrahedra that are on the same side of triangulation $\mathcal{T}$.

3 C^2 Mesh of Quintic Patches

The input of our algorithm is a triangulated polyhedron $\mathcal{P}$, with optional C^2 data at each vertex. C^2 data means that each vertex p is given a gradient vector $\Delta f(p)$ and a Hessian matrix $\Delta^2 f(p)$. The gradient vector and/or Hessian matrix can be specified by a plane or a conic surface $F_p(x,y,z) = 0$ goes through p,

For each vertex that has incomplete C^2 data, a preprocessing step computes reasonable ones by fitting a conic surface to the vertex and its neighboring vertices.

We then construct a simplicial hull $\sum$ as in [BCX95a], except that we impose a stronger tangent containment condition: For a tetrahedral simplex $[p_1p_2p_3p_4]$ built on polyhedral face $[p_1p_2p_3]$,

$$|N_{p_1} \cdot [p_1p_4]| > \zeta, \tag{3.14}$$

where N_{p_1} is the normal at p_1 and ζ is a given small positive constant. We will show later that we need this stronger condition to guarantee that piecewise surface is connected within Σ. This stronger condition can usually be enforced by raising p_4.

Once we have established a simplicial hull $\sum$ for the given polyhedron $\mathcal{P}$ and a set of point normals N, we construct a C^2 trivariate piecewise polynomial function F within $\sum$ such that F has the given C^2 data at each vertex. We then further set the free polynomial coefficients so that they are C^2 continuous to each other. We adapted the C^1 and functional cubic scheme from [BCX95a] to quintic piecewise polynomial. Please refer to it for a full description of the C^1 scheme.

Figure 3.3 illustrates between the weights of neighboring tetrahedral simplex in Σ.

$$\begin{array}{llll} V_1 = [p_1p_2p_3p_4], & V_2 = [p_1'p_2p_3p_4'], & W_1 = [p_1''p_2p_3p_4], & W_2 = [p_1''p_2p_3p_4'], \\ V_1' = [p_1p_2p_3q_4], & V_2' = [p_1'p_2p_3q_4'], & W_1' = [q_1''p_2p_3p_4], & W_2' = [q_1''p_2p_3p_4'], \end{array}$$

and the polynomials f_i over V_i, g_i over W_i, f_i' over V_i' and g_i' over W_i' be expressed in Bernstein-Bezier forms with coefficients a_λ^i, b_λ^i and c_λ^i, respectively. Now we shall determine these coefficients

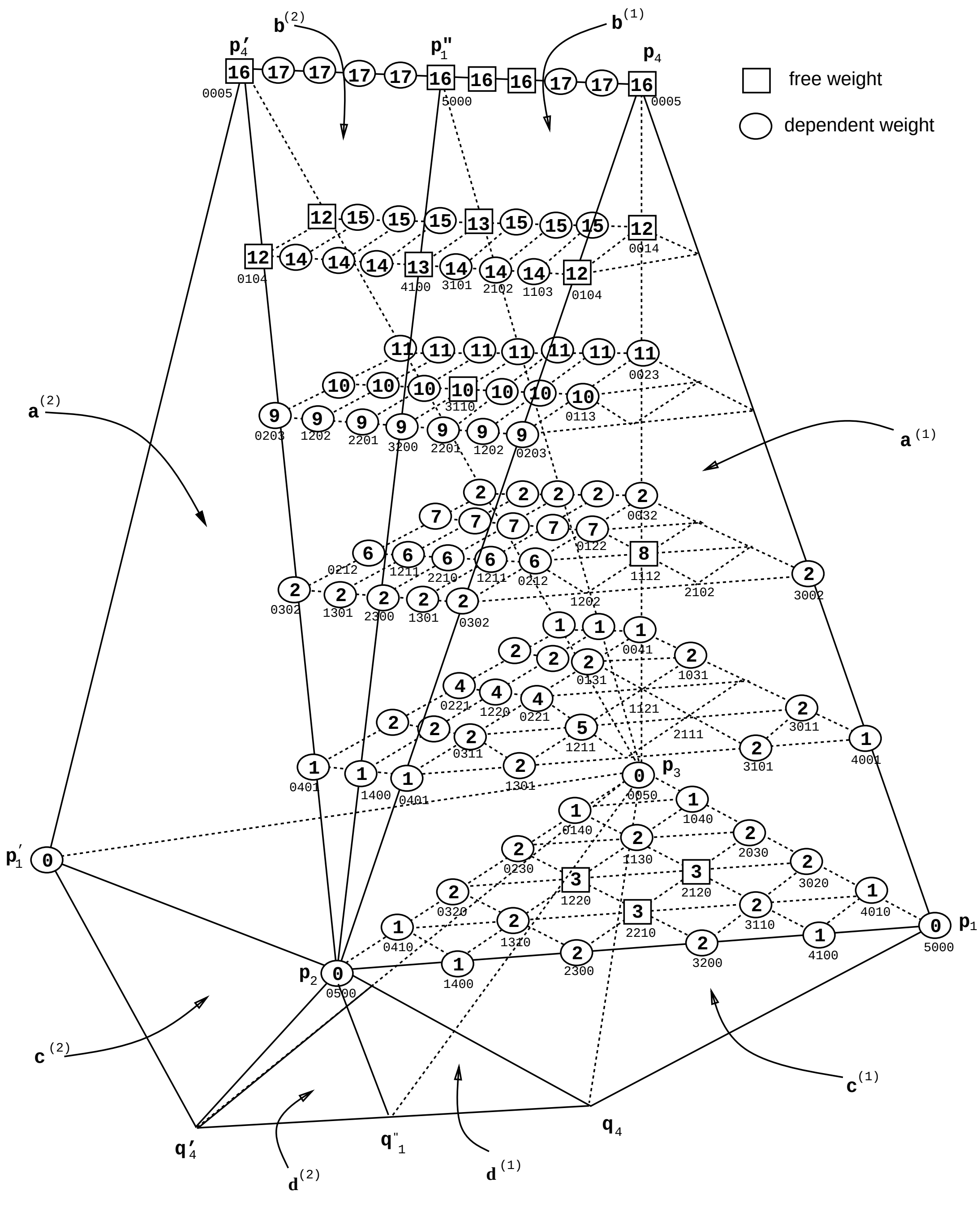


Figure 3.3: Adjacent Tetrahedra, Quintic Functions and Control Points for two Non-Convex Adjacent Faces

step by step. Denote

$$\begin{array}{rcll} p_1'' & = & \beta_1^1 p_1 + \beta_2^1 p_2 + \beta_3^1 p_3 + \beta_4^1 p_4, & \beta_1^1 + \beta_2^1 + \beta_3^1 + \beta_4^1 = 1 \\ p_1'' & = & \beta_1^2 p_1' + \beta_2^2 p_2 + \beta_3^2 p_3 + \beta_4^2 p_4', & \beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 = 1 \\ p_1'' & = & \mu_1 p_4 + \mu_2 p_4', & \mu_1 + \mu_2 = 1 \\ q_1'' & = & \gamma_1 q_4 + \gamma_2 q_4', & \gamma_1 + \gamma_2 = 1 \end{array} \tag{3.15}$$

Algorithm 3.1 C^2 *quintic scheme(see Figure 3.3)*

1. The number 0 weights are given by the function values at the vertices. For examples, $a_{5e_i}^{(1)} = f(p_i),\ i = 1, 2, 3$.
2. The number 1 weights are determined by formula (2.7).
3. The number 2 weights are determined by formula (2.8).
4. The number 3 weights, that is $a_{1220}^{(i)}, a_{2210}^{(i)}$ and $a_{2120}^{(i)}$, are free.
5. The number 4 weights are determined by C^1 condition (2.10).
6. The number 5 and 6 weights have to be determined simultaneously. In determining these weights, we need to consider all the C^1 and C^2 conditions related to the tetrahedra surrounding the vertex p_2. Suppose there are k triangles(hence k edges) around p_2.

   Along edge $[p_2p_3]$, there are a "clique" of 8 sets of weights with respect of the 8 tetrahedra around $[p_2p_3]$. $W_{2210}^{(i)}, W_{1310}^{(i)}, W_{0410}^{(i)}, W_{1220}^{(i)}, W_{0320}^{(i)}, W_{0230}^{(i)}, W_{0311}^{(i)}, W_{1211}^{(i)}, W_{0221}^{(i)}, W_{0212}^{(i)}, i = 1, 2, W = a, b, c, d$. Actually each of the 8 sets is just a different BB-form with respect to a different basis. Further studying their relations, we figure that there is a linear relation between $a_{1211}^{(1)}$ and $a_{1211}^{(2)}$, regardless of the number 6 weights or weights $c_{1211}^{(i)}$. However, a number 5 weight, for example a_{1211}^1, is shared by two "cliques" around $[p_2p_3]$ and $[p_2p_1]$. Therefore, around vertex p_2, there is a $k \times k$ linear system. One can show that this system is of rank $k-1$ and hence always solvable. $a_{1211}^{(i)}$ are set by solving the system. The other number 5 weights and number 6 weights are set by C^2 condition. Please note that the symmetric linear system with respect to $c_{1211}^{(i)}$ on the other side is trivially satisfied by the solution of $a_{1211}^{(i)}$. See Appendix A for further details.
7. The number 7 weights are similarly determined as that of number 6 weights.
8. The number 8 weight $a_{1112}^{(i)}$ are free.
9. The number 9 weights are determined by C^1 and C^2 conditions. Both the number of equations and the number of unknowns are $6k$. See Appendix B for details.
10. For the number 10 weights, we have six equations parallel to the equations (B.53)–(B.56) with all the index changed by the rule

    $$\text{the index of the number 10 weright} = \text{the index of the number } 9 - e_2 + e_3 \tag{3.16}$$

    and seven independent weights. By choosing one of them, say $b_{3110}^{(i)}$, to be a free parameter, the entire system can be solved.

11. The number 11 weights are determined in the same way as the that of number 9 weights.

12. The number 12 and 13 weights are free. The number 14 weights are determined by C^1 and C^2 conditions. That is $b^{(i)}_{1103}$ are defined by (2.10), and $b^{(i)}_{2102}$ are defined by (2.11). For $b^{(i)}_{3101}$, we have by (2.12) and (2.13)

$$\mu_1 b^{(1)}_{3101} + \mu_2 b^{(2)}_{3101} = b^{(1)}_{4100}$$

$$-\mu_1 b^{(1)}_{3101} + \mu_2 b^{(2)}_{3101} = \mu_2^2 b^{(2)}_{2102} - \mu_1^2 b^{(1)}_{2102}$$

Hence

$$b^{(1)}_{3101} = \frac{b^{(1)}_{4100} - \mu_2^2 b^{(2)}_{2102} + \mu_1^2 b^{(1)}_{2102}}{2\mu_1}$$

$$b^{(2)}_{3101} = \frac{b^{(1)}_{4100} + \mu_2^2 b^{(2)}_{2102} - \mu_1^2 b^{(1)}_{2102}}{2\mu_2}$$

13. The number 15 weights are similar to that of number 14, the index is changed by the rule (3.16).

14. The number 16 weights are free, the number 17's are determined by C^1 and C^2 conditions.

15. The remaining weights with index $\lambda_1\lambda_2\lambda_3\lambda_4$ are determined by C^1 and C^2 conditions (2.10) and (2.11) for $\lambda_4 \le 2$ and freely chosen for $\lambda_4 > 2$

In summary, the construction steps 1–14 are according to the C^1 and C^2 conditions across the common tetrahedra faces that are over or below the original triangulation. Step 15 is according to the C^1 and C^2 conditions across the common tetrahedra faces that are on the original triangulation. Therefore, the composite polynomial function is global C^2 continuous.

In the algorithm, there are quite some free weights need to be set. whatever values they are assigned to have impacts on the shape of the surface. Instead of an optimization analysis, we proposed two kinds of cheap heuristics. The first method is the following. We first construct a C^1 A-patch surface with cubic patches. After we make sure that the surface looks good, we degree-raised the whole simplicial hull to quintic patches. Now whatever weights a quintic patch has will be used as default values if needed. The second method is to always set the free weight so that the polynomial approximate a polynomial of lower degree. We observe that the first method works better.

4 Functional Zero Contouring Quintics

This section gives a set of sufficient conditions to ensure that each quintic is an A-patch. Namely, besides C^2 continuous between adjacent tetrahedra, the zero contouring surface is singly-connected, non-singular and topologically equivalent to $\mathcal{T}$.

In the C^1 cubic scheme, one is able to arrange the weights of the BB polynomial so that there is a layer of weights that separates the nonpositive weights and nonnegative. Such an arrangement turns out to be a simple sufficient condition for an A-patch. However, in the C^2 quintic scheme, after meeting the C^2 conditions, the signs of the weights could alternate through the the layers. In particular, the weights at the 1st, 2nd and 3rd layers can be of any sign. We show in the following

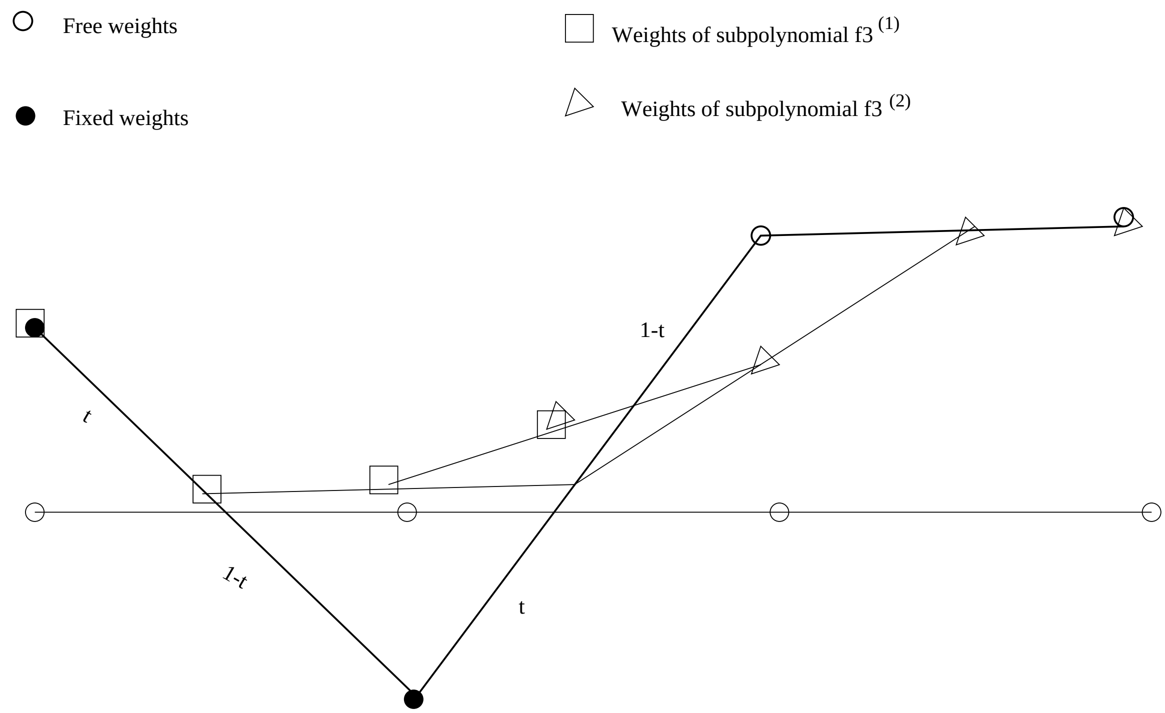


Figure 4.4: Making sure a cubic polynomial is positive by looking into its two subpolynomials

text that, one can arrange the 4th, 5th layers to be nonnegative and large enough in absolute value, so that the zero contouring surface is an A-patch.

We first discuss some sufficient conditions to guarantee that a BB polynomial is positive in its domain. Then we discuss some sufficient conditions to make sure a univariate BB polynomial has exactly one root between $(0, 1)$. Then we show that the sufficient functional zero contouring conditions for a quintic is an integration of the single-rooted and non-rooted conditions at every rays, for a three-sided patch, shooting from the top vertex p_4 of the tetrahedron to the bottom face, or for a four sided patch, from edge $[p_4, p_1'']$ of the tetrahedron to edge $[p_2, p_3]$.(for a four See Figure 3.3.

4.1 Enforcing positivity of a BB polynomial

4.1.1 Univariate

A necessary condition for a univariate polynomial $F(\alpha)$ to be positive in $[0, 1]$ is that $F(0) = b_{m0} > 0$ and $F(1) = b_{0m} > 0$. Actually, if we are given $F(0, 1) = b_{m0} > \varepsilon$, for some small constant $\varepsilon > 0$, and some other weights excluding $F(1, 0) = b_{0m}$, we can always set b_{0m} and other free weights to be large enough so that $F(\alpha) > 0, \alpha \in [0.1]$.

In stead of giving a full proof, we check out several cases that are going to be used in weight setting processes.

For a cubic BB polynomial

$$F3(\alpha) = b_{30}B^3_{30}(\alpha) + b_{21}B^3_{21}(\alpha) + b_{12}B^3_{12}(\alpha) + b_{03}B^3_{03}(\alpha) \tag{4.17}$$

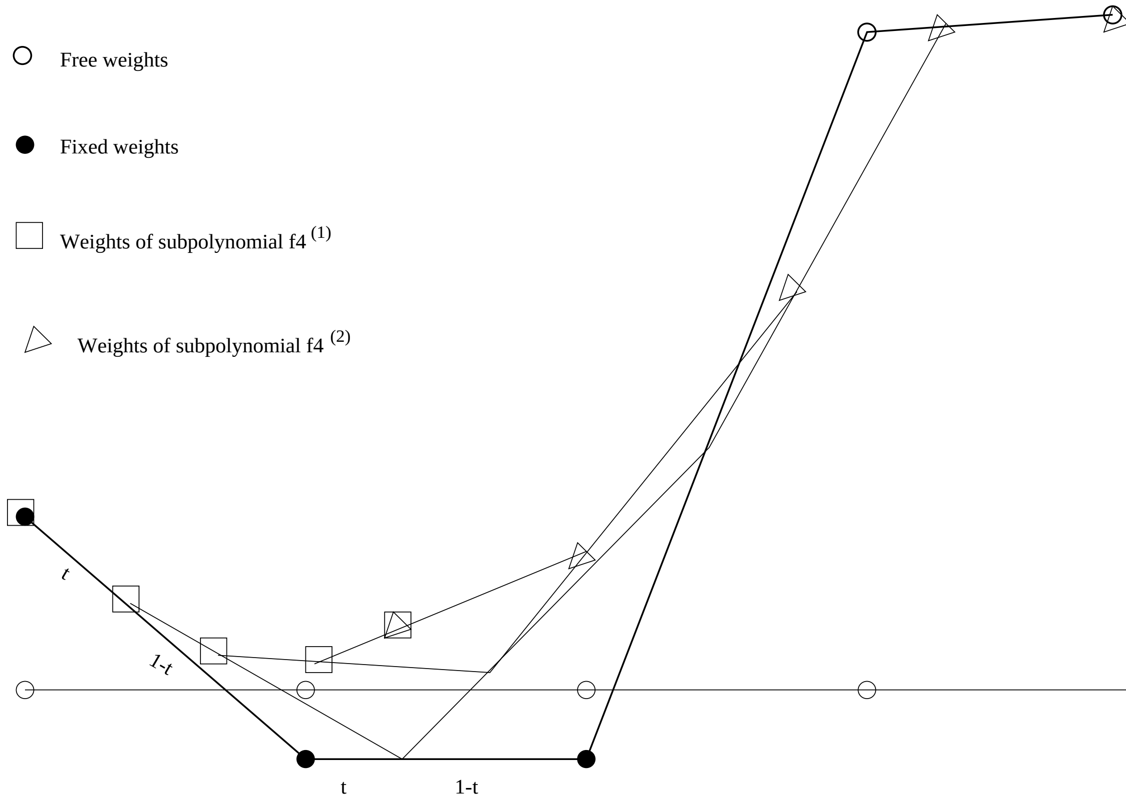


Figure 4.5: Making sure a quartic polynomial is positive by looking at its two subpolynomials

where $b_{30} > 0$, b_{21} are given, set b_{12} and b_{03} so that $F3(\alpha) > 0$ in $[0, 1]$.

From the subdivision properties of BB polynomial, if $F3(\alpha) > 0$, then there must exist a subdivision of the BB polynomial where the weights of each piece are all positive. Based on this observation, we subdivide the univariate cubic polynomial $F3(\alpha)$ into two pieces,

$$F3^{(1)} = b_{30}^{(1)}B_{30}^3(\alpha) + b_{21}^{(1)}B_{21}^3(\alpha) + b_{12}^{(1)}B_{12}^3(\alpha) + b_{03}^{(1)}B_{03}^3(\alpha) \quad (4.18)$$

$$F3^{(2)} = b_{30}^{(2)}B_{30}^3(\alpha) + b_{21}^{(2)}B_{21}^3(\alpha) + b_{12}^{(2)}B_{12}^3(\alpha) + b_{03}^{(2)}B_{03}^3(\alpha) \quad (4.19)$$

upon $\alpha = (1-t, t)$ (See figure 4.4). We now just need to choose t carefully so that

$$b_{21}^{(1)} = (1-t)b_{30} + tb_{21} > 0 \quad (4.20)$$

The other weights are positive function of b_{12} and b_{03} and hence can be set to be positive by setting b_{12} and b_{03} large enough. Quantitative details are given in appendix C.

For a quartic BB polynomial,

$$F4(\alpha) = b_{40}B_{40}^4(\alpha) + b_{31}B_{31}^4(\alpha) + b_{22}B_{22}^4(\alpha) + b_{13}B_{13}^4(\alpha) + b_{04}B_{04}^4(\alpha) \quad (4.21)$$

where $b_{40} > \varepsilon$, b_{31} and b_{22} are given, set b_{13} and b_{04} so that $F4(\alpha) > 0$ in $[0, 1]$.

We subdivide the univariate quartic polynomial $F4(\alpha)$ into two pieces,

$$F4^{(1)}(\alpha) = b_{40}^{(1)}B_{40}^4 + b_{31}^{(1)}B_{31}^4 + b_{22}^{(1)}B_{22}^4 + b_{13}^{(1)}B_{13}^4 + b_{04}^{(1)}B_{04}^4 \quad (4.22)$$

$$F4^{(2)}(\alpha) = b_{40}^{(2)}B_{40}^4 + b_{31}^{(2)}B_{31}^4 + b_{22}^{(2)}B_{22}^4 + b_{13}^{(2)}B_{13}^4 + b_{04}^{(2)}B_{04}^4 \quad (4.23)$$

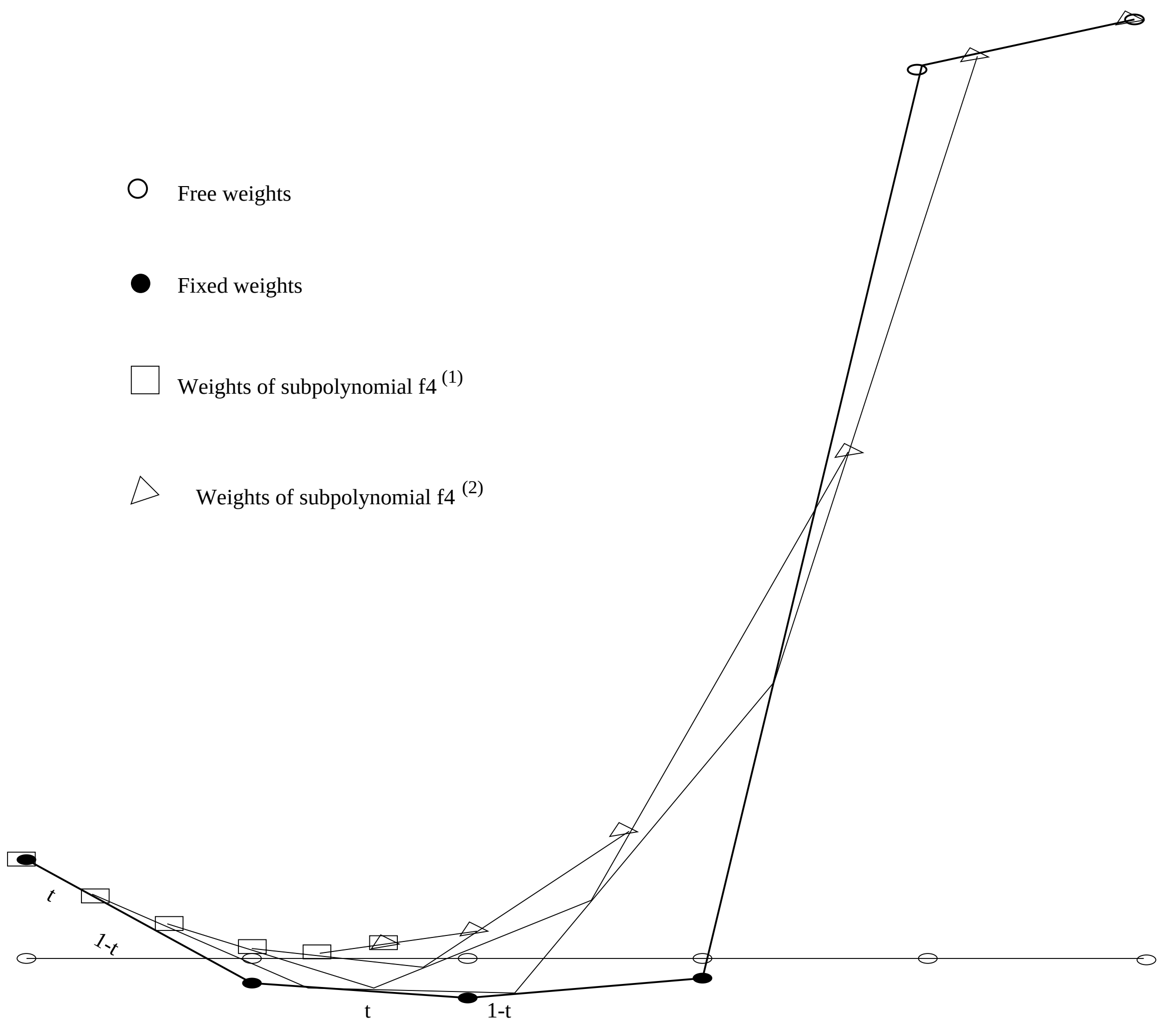


Figure 4.6: Making sure a quintic polynomial is positive by looking at its two subpolynomials

upon $\alpha = (1-t, t)$ (see figure 4.5). By choosing t carefully, one makes sure that

$$b_{31}^{(1)} = (1-t)b_{40} + tb_{31} > 0 \tag{4.24}$$
$$b_{22}^{(1)} = (1-t)^2 b_{40} + 2(1-t)tb_{31} + t^2 b_{22} > 0. \tag{4.25}$$

The rest of the control points is a positive function of b_{13}, b_{04} and hence can be set to be positive. Quantitative details are given in appendix C

For a quintic BB polynomial,

$$F5(\alpha) = b_{50}B_{50}^5(\alpha) + b_{41}B_{41}^5(\alpha) + b_{32}B_{32}^5(\alpha) + b_{23}B_{23}^5(\alpha) + b_{14}B_{14}^5(\alpha) + b_{05}B_{05}^4(\alpha) \tag{4.26}$$

where $b_{50} > \varepsilon$, b_{41}, b_{32} and b_{23} are given, set b_{14} and b_{05} so that $F5(\alpha) > 0$ in $[0,1]$.

We subdivide the univariate quintic polynomial $F5(\alpha)$ into two pieces,

$$F5^{(1)}(\alpha) = b_{50}^{(1)}B_{50}^5 + b_{41}^{(1)}B_{41}^5 + b_{32}^{(1)}B_{32}^5 + b_{23}^{(1)}B_{23}^5 b_{14}^{(1)}B_{14}^5 + b_{05}^{(1)}B_{05}^5 \tag{4.27}$$
$$F5^{(2)}(\alpha) = b_{50}^{(2)}B_{50}^5 + b_{41}^{(2)}B_{41}^5 + b_{32}^{(2)}B_{32}^5 + b_{23}^{(2)}B_{23}^5 b_{14}^{(2)}B_{14}^5 + b_{05}^{(2)}B_{05}^5 \tag{4.28}$$

upon $\alpha = (1-t, t$ (see figure 4.6). By choosing t carefully, one makes sure that

$$
\begin{aligned}
b_{41}^{(1)} &= (1-t)b_{50} + tb_{41} > 0 && (4.29)\\
b_{32}^{(1)} &= (1-t)^2 b_{50} + 2(1-t)tb_{41} + t^2 b_{32} > 0 && (4.30)\\
b_{23}^{(1)} &= (1-t)^3 b_{50} + 3(1-t)^2 tb_{41} + 3(1-t)t^2 b_{32} + t^3 b_{23} > 0 && (4.31)
\end{aligned}
$$

The rest of the control points is a positive function of b_{14}, b_{05} and hence can be set to be positive. Quantitative details are given in appendix C

4.1.2 Bivariate

Given a bivariate degree m BB polynomial $F(\alpha)$ in $[p_1p_2p_3]$, if $F(\alpha) \wedge \alpha_3 = 0$ is fixed and larger than a small constant ε, we can set b_{00m} and any other free weights so that $F(\alpha) > 0$ in the domain.

A sufficient algorithm to achieve this is given as follows. We first construct a "worst case" univariate degree m polynomial

$$Fm^-(\alpha) = b_{m0}^- B_{m0}^m(\alpha) + ... + b_{(m-j)j}^- B_{(m-j)j}^m(\alpha) + b_{0m}^- B_{0m}^m(\alpha). \tag{4.32}$$

where, for the jth layer $F(\alpha) \wedge \alpha_3 = j$, $0 < j < m$, if a minimum value can be determined from the fixed weights assign it to $b_{(m-j)j}^-$. Otherwise mark $b_{(m-j)j}^-$ as a free weights and layer j as free layer. We then set the free weights to make sure the univariate polynomial $Fm^- > 0$. Then we make sure that the subpolynomial in each free layer is larger than the corresponding free weight in Fm^-. By doing that, we make sure that the univariate subpolynomial in any line $[pp_3]$, $p \in [p_1, p_2]$, is positive.

If we have more time or our computer is more powerful, we may obtain more relax result by first subdividing $[p_1p_2]$ into n pieces, $[p_1^{(i)}, p_2^{(i)}]$, $p_1^{(1)} = p_1$, $p_2^{(n)} = p_2$. Then applying the above algorithm to each triangular simplex $[p_1^{(i)}, p_2^{(i)} p_3]$, and then in each layer, setting the free weights so that it is no less than what is required for each subdivided subpolynomial.

Similarly, if b_{00m} is fixed and larger than a small constant ε, we can set the weights of $F(\alpha) \wedge \alpha_3 = 0$ and any other free weights so that $F(\alpha) > 0$ in the domain.

4.1.3 Trivariate

There are two kinds of configurations with trivariate BB polynomials. The first case is analog to the bivariate case. For a trivariate degree m BB polynomial $F(\alpha)$ in $[p_1p_2p_3p_4]$, if $F(\alpha) \wedge \alpha_4 = 0$ is fixed and larger than a small constant ε, we can always set b_{000m} and any other free weights so that $F(\alpha) > 0$ for any α over the domain simplex. If $b_{000m} > \varepsilon > 0$ is given, we can always set the weights of subpolynomial in $[p_1p_2p_3]$, namely $F(\alpha) \wedge (\alpha = 0)$ and any other free weights to be large enough so that $F(\alpha) > 0$.

The second case, if $F(\alpha) \wedge \alpha_1 = \alpha_4 = 0$ is fixed and larger than a small constant ε, we can always set the weights of $F(\alpha) \wedge \alpha_2 = \alpha_3 = 0$ and any other free weight to be large enough so that $F(\alpha) > 0$. In particular, the worst case polynomial

$$Fm^-(\alpha) = b_{m0}^- B_{m0}^m(\alpha) + ... + b_{(m-j)j}^- B_{(m-j)j}^m(\alpha) + + ... + b_{0m}^-(\alpha) \tag{4.33}$$

where $b_{(m-j)j}^-$ is the minimum of $F(\alpha) \wedge \alpha_1 + \alpha_4 = j$, if the minimum can be determined. Otherwise $b_{(m-j)j}^-$ is marked free.

Summing up, in a simplex tetrahedron, if the subpolynomial in a face is given positive, we can adjust the subpolynomial at its opposite vertex so that the entire polynomial over the simplex is positive and vice versa. If the subpolynomial in an edge is given positive, we can adjust that subpolynomial at its opposite edge so that the entire polynomial over the simplex is positive. The first case is for the three sided patch while the second case is for the four sided case.

Similarly, relax result can be obtained by subdividing $[p_1p_2p_3p_4]$. For the first case (three sided patch), $[p_1p_2p_3p_4]$ is subdivided into $[p_1^{(i)}p_2^{(i)}p_3^{(i)}p_4]$'s, where $[p_1^{(i)}p_2^{(i)}p_3^{(i)}]$ is a sub-triangle of $[p_1p_2p_3]$. For the second case (four sided patch), $[p_1p_2p_3p_4]$ is subdivided into $[p_i^{s1}p_{i+1}^{s1}p_i^{s2}p_{i+1}^{s2}$'s, where $[p_i^{s1}p_{i+1}^{s1}$ is a subsegment of $[p_1p_4]$, and $[p_i^{s2}p_{i+1}^{s2}$ is a subsegment of $[p_2p_3]$.

4.2 Sufficient conditions for single-rooted univariate quintic BB polynomials

We now tackle the following problem. Give a univariate quintic BB polynomial

$$F5(\alpha) = b_{50}B_{50}^5(\alpha) + b_{41}B_{41}^5(\alpha) + b_{32}B_{32}^5(\alpha) + b_{23}B_{23}^5(\alpha) + b_{14}B_{14}^5(\alpha) + b_{05}B_{05}^5(\alpha) \tag{4.34}$$

where b_{50}, b_{41}, b_{32}, and b_{23} are given, set b_{14} and b_{05} so that the polynomial has exactly one root within the interval of $(0,1)$ if (1) $b_{50} < 0$, or (2) $b_{50} = 0$ and $b_{41} > \eta$, where $\eta > 0$ is a constant.

We first assume that $b_{50} < 0$. Such a quintic BB polynomial can be classified into the 8 cases in terms of the signs of b_{41}, b_{32} and b_{23}, shown in figure 4.7. If we denote each case by a triple consists of the signs of b_{41}, b_{32} and b_{23} respectively, then in cases $(-,-,-)$, $(-,-,+)$, $(-,+,+)$ and $(+,+,+)$, there is only one sign change in the weights, which is a sufficient condition to guarantee that there is exactly one root in $[0,1]$. We call these four cases category (0).

The other four cases are divided into the following two categories in terms of the difference between b_{50} and b_{41}: (1) $b_{50} - b_{41} < -\varepsilon$, where ε is a small positive number, which can be set as $b_{50}/100$; and (2) otherwise. Case $(+,+,-)$, $(+,-,+)$ and $(+,-,-)$ fall into case (1), while case $(-,+,-)$ is further divided into (1) and (2) (See figure 4.8).

For a polynomial of category (1), we set b_{14} and b_{05} to make sure that the first derivative of $F5(\alpha)$,

$$\begin{aligned} F5_\alpha(\alpha) &= 5((b_{41} - b_{50})B_{40}^4(\alpha) + (b_{32} - b_{41})B_{31}^4(\alpha) + (b_{23} - b_{32})B_{22}^4(\alpha) + \\ &\quad (b_{14} - b_{23})B_{13}^4(\alpha) + (b_{05} - b_{14})B_{04}^4(\alpha)) \end{aligned} \tag{4.35}$$

is positive in $[0,1]$. It follow that $F5(\alpha)$ is monotonic and hence single-rooted. The problem is hence reduced to the quartic case we have discussed in last subsection.

For a polynomial of case (2), we set b_{14} and b_{05} to make sure that the second derivative of $F(\alpha)$,

$$\begin{aligned} F5_{\alpha\alpha}(\alpha) &= 20((b_{32} - 2b_{41} + b_{50})B_{30}^3(\alpha) + (b_{23} - 2b_{32} + b_{41})B_{21}^3(\alpha) + \\ &\quad (b_{14} - 2b_{23} + b_{32})B_{12}^3(\alpha) + (b_{05} - 2b_{14} + b_{23})B_{03}^3(\alpha)) \end{aligned} \tag{4.36}$$

is positive. The positivity of $F_{\alpha\alpha}(\alpha)$ ensure that $F_\alpha(\alpha)$ is monotonic, which ensure that $F(\alpha)$ is single-rooted, providing that $F5(0) = b_{50} < 0$ and $F5(1) = b_{05} > 0$. As $b_{50} < 0$, $b_{41} < 0$, $b_{32} > 0$ and $b_{50} - b_{41} \le \varepsilon$ we have $b_{32} - 2b_{41} + b_{50} > 0$. Hence the problem is reduced to the cubic case we discuss in last subsection.

In the case that b_{50} and $b_{41} > \eta > 0$, the polynomial has a root at $\alpha = (1,0)$. As $b_{41} > \eta > 0$, we can set b_{14} and b_{05} large enough so that there is no other roots.

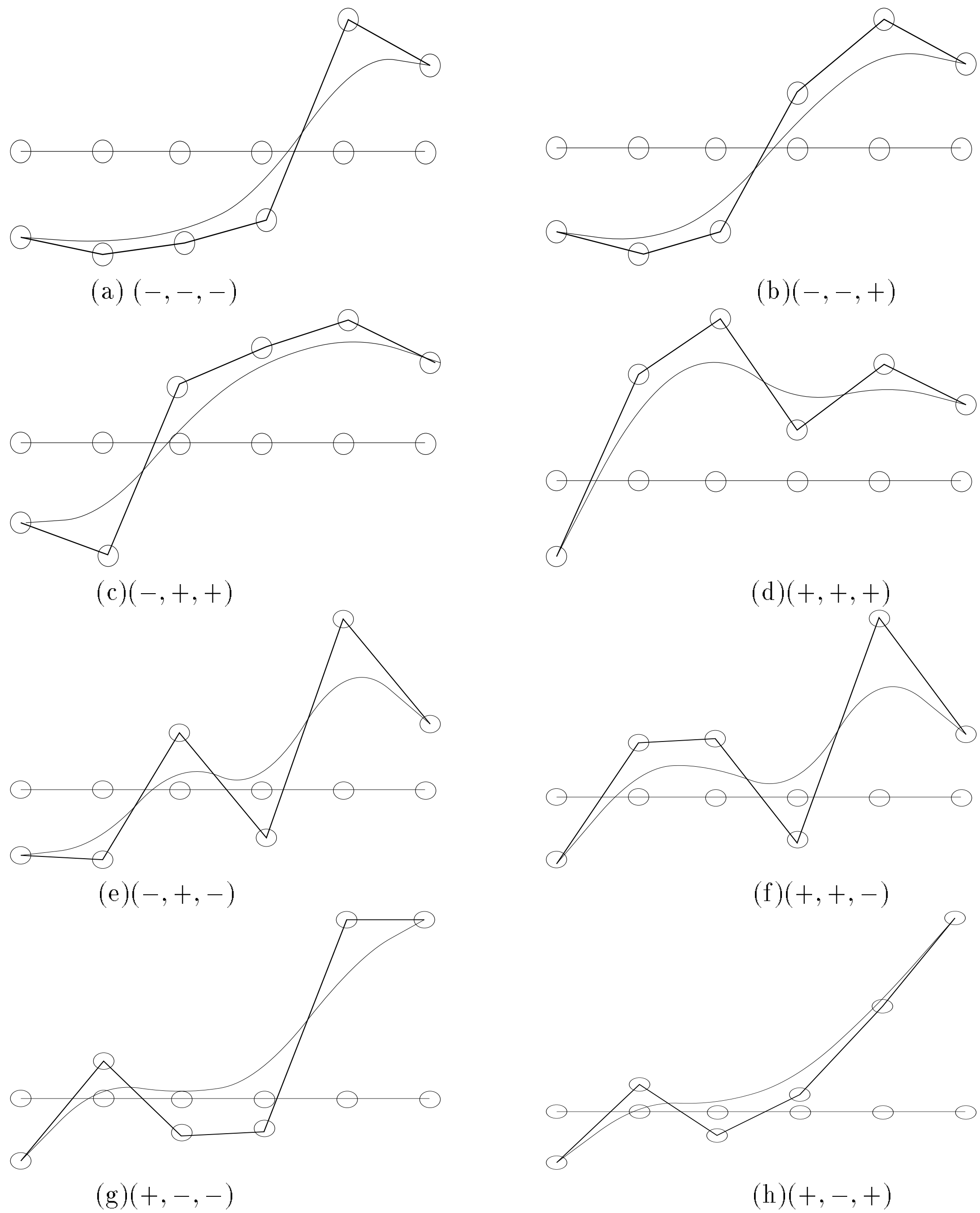
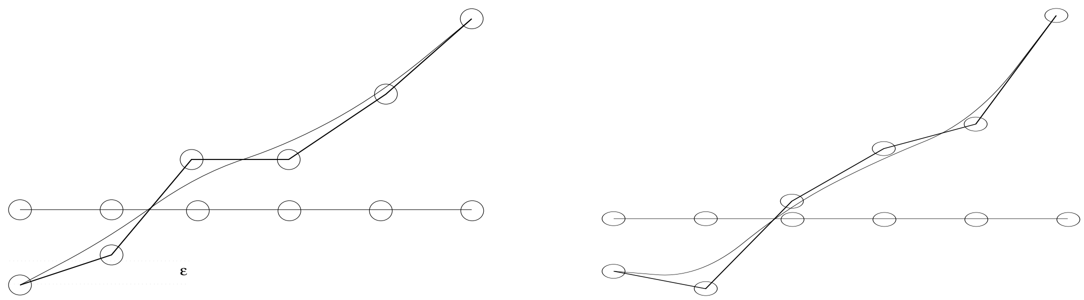


Figure 4.7: Quintic BB polynomials are subdivided into 8 cases according to the signs of weights b_{41}, b_{32} abd b_{23}.



(a) (b)

Figure 4.8: case $(-,+,-)$ is further subdivided. (a) $b_{50} - b_{41} < -\varepsilon$. (b) Otherwise

4.3 Sufficient conditions for quintic A-patches

From definition, a three sided quintic patch is an A-patch if a line segment $[pp_4]$ connecting the top vertex p_4 and any point p on the bottom $[p_1p_2p_3]$ intersects the surface at most once. In other words, the univariate subpolynomial in $[pp_4]$ has at most one root in $[0,1]$. In the case of a four sided patch, a *ray polynomial* is the univariate subpolynomial in $[p_{14}p_{23}]$, where $p_{14} \in [p_1p_4]$, $p_{23} \in [p_2p_3]$.

We first assume that, at the bottom face $[p_1p_2p_3]$ (where $\alpha_4 = 0$), $F(\alpha) > 0$. Then $[pp_4]$ intersects the surface exactly once. We call such a subpolynomial a *ray polynomial* of the trivariate polynomial in $[p_1p_2p_3p_4]$. Note that how many times $[pp_4]$ intersects with the surface are determined by the ray polynomial in $[pp_4]$. Recalling that, in section 3, the weights at the 4th and 5th are free in the C^2 continuity weight setting algorithm, each of such univariate quintic polynomials can be made single-rooted as we discussed above. So if one can check all $[p_4p]$'s for all p in $[p_1p_2p_3]$, to make sure that the ray polynomial has exactly one root in $[0,1]$, we make sure that the surface is an A-patch. But obviously, it is non-practical to keep track of the infinite number of ray polynomials. Hence, we instead subdivide the bottom face $[p_1p_2p_3]$ into small triangles $[p_1^{(i)}p_2^{(i)}p_3^{(i)}]$, such that the subpolynomial in $[p_1^{(i)}p_2^{(i)}p_3^{(i)}p_4]$ is simpler in the sense that the ray polynomials fall in the same category that we discussed above. We are then able to make sure that all ray polynomials in $[p_4p]$ are single-rooted by making sure that the "worst case" is single-rooted.

Similarly, in the case of a four sided patch $[p_1p_2p_3p_4]$, we subdivide edge $[p_1p_4]$ and edge $[p_2p_3]$ to subdivide the tetrahedron into smaller four sided patches, within each of which we enforce that the surface is an A-patch by treating a "worst" case ray polynomial.

However, it is not practical to subdivide the polynomial into into subpolynomials that exactly fall into individual categories. Instead, we subdivide the polynomial until the individual subpolynomial can be treated the same way. To see that is possible, in last subsection, we classify the univariate quintic subpolynomials into 3 categories. Category (0) can be trivially made single-sheeted by setting b_{14} and b_{05} to be positive. So, if a ray polynomial of a trivariate polynomial is either in category (0) or (1), we can treat them all as if they are all in category (1). Similarly, when a ray polynomial of a trivariate polynomial is either in category (0) or (2). The algorithm that guarantees a category (2) subpolynomial single-rooted rely on the fact that $b_{50} + b_{32} - 2b_{41} > 0$, which can apply to some member of categories (0) and (1) as well. Hence if we subdivide the polynomial deep enough, we can get subpolynomial whose ray polynomials are in category (2), or (0), or (1) but with $b_{50} + b_{32} - 2b_{41} > 0$.

Based on the above observation, we layout the subdivision scheme in the following way.

Let $F_0(\alpha) = F(\alpha) \wedge (\alpha_4 = 0)$ be the bivariate polynomial in the bottom face, or the 0th layer, $F_1(\alpha) = F(\alpha) \wedge (\alpha_4 = 1)$ be the quartic bivariate polynomial in the 1st layer, $F_2(\alpha) = F(\alpha) \wedge (\alpha_4 = 2)$ be the cubic bivariate polynomial in the 2nd layer. We define the following two polynomials in $[p_1 p_2 p_3 p_4]$

$$F_\Delta(\alpha) = F_0(\alpha) - F_1(\alpha)\alpha \tag{4.37}$$

$$F_{\Delta^2}(\alpha) = F_0(\alpha) - 2F_1(\alpha)\alpha + F_2(\alpha)\alpha^2 \tag{4.38}$$

We denote the weights of polynomial $F(\alpha)$ as $\mathbf{b}(F)$. $\mathbf{b}(F) < c$ means the weights of F are all less than c.

Then we keep subdividing $[p_1 p_2 p_3 p_4]$ until for each subpolynomial $[p_1^{(i)} p_2^{(i)} p_3^{(i)} p_4]$ satisfies one of the following cases.

(0) There is only one sign change;

(1) $\mathbf{b}(F_\Delta) < -\varepsilon$, which implies that for every univariate subpolynomial in $[p_4 p]$, $b_{50} - b_{41} < -\varepsilon$;

(2) $\mathbf{b}(F_1) < 0 \wedge (\mathbf{b}(F_{\Delta^2}) \vee \mathbf{b}(F_2) < 0 \vee \mathbf{b}(F_3) > 0)$ which implies that for every univariate subpolynomial in $[p_4 p]$, $b_{41} < 0$ and $b_{50} - 2b_{41} + b_{32} > 0$, or can be classified into category (0).

In cases (1) and (2), we define the *worst* univariate subpolynomial

$$F_{worst}(x) = b_{50}^- B_{50}^5 + b_{41}^- B_{41}^5 + b_{32}^- B_{32}^5 + +b_{23}^- B_{23}^5 + b_{41}^- B_{14}^5 + b_{50}^- B_{05}^5 \tag{4.39}$$

where

$$\begin{aligned} b_{50}^- &= Min\{\mathbf{b}(F_0)\} \\ b_{41}^- &= Min\{\mathbf{b}(F_1)\} \\ b_{32}^- &= Min\{\mathbf{b}(F_2)\} \end{aligned}$$

Enforcing the worst univariate subpolynomial to be single-rooted, we guarantee that all the univariate subpolynomial over $[p_4 p]$ is single-rooted, which implies that the surface is an A-patch.

In the case that $F_0(\alpha)$ is not always positive, we subdivide face $[p_1 p_2 p_3]$ until in each subpolynomial $F_0^{(i)}(\alpha)$ in $[p_1^{(i)} p_2^{(i)} p_3^{(i)}]$ is one of the following

1. Positive. Set the free weights of $F^{(i)}(\alpha)$ so that the trivariate polynomial has no root.

2. Negative. Set the free weights of $F^{(i)}(\alpha)$ so that the trivariate polynomial has exactly one root.

3. The subpolynomial in $[p_1^{(i)} p_2^{(i)} p_3^{(i)}]$ (denoted as $F_4^{(i)}(\alpha)$) is an A-spline [BX92]. A sufficient condition is that, there exists an integer $j (1 \le j \le 3)$ an integer $k (0 < k < 5)$ such that

$$b_\lambda \le 0, \qquad \lambda_j = 0, 1, ..., k-1 \tag{4.40}$$

$$b_\lambda \ge 0, \qquad \lambda_j = k+1, ..., 5 \tag{4.41}$$

   In this case, say $j = 1$, we set the free weights of $F^{(i)}(\alpha)$ so that the bivariate subpolynomial in $[p_2^{(i)} p_3^{(i)} p_4]$ is positive, and furthermore, large enough so that any ray subpolynomial in pp_4, where $p \in [p_2^{(i)} p_3^{(i)} p_4]$, is single-rooted.

At a corner of face $[p_1p_2p_3]$, say p_1 or $\alpha = (1,0,0,0)$, the weight $b_{5000} = 0$. From the stronger tangent containment condition 3.14 that we have enforced, $b_{4001} > \eta$ for some small constant $\eta > 0$. Thus there exists a small neighborhood $\alpha^*, |\alpha^* - \alpha| < \rho$, where $F_1(\alpha^*) > 0$. Hence it can be treated as an instance of category (1).

In short, the functional zero contour enforcing scheme can be embed into the C^2 scheme as follows. Steps 4 and 8 replace steps 4 and 8 in scheme 3.1. Step 11' is inserted between steps 11 and 12.

Algorithm 4.2 *Functional zero contouring*

4. If face $[p_1p_2p_3]$ is convex, set $a^{(i)}_{1220}$, $a^{(i)}_{2210}$, $a^{(i)}_{2120}$ to be positive.

8. Set number 8 weight, $a^{(i)}_{1112}$ to be positive.

11'. Set number 12 weight to be large enough so that

- **(a)** we are sure the polynomial in $[p_1p_2p_3p_4]$ is a three sided A-patch, and
- **(b)** when we compute number 13, 14 and 15 weights later in step 12, they are large enough so that we are sure the polynomial in $[p_1''p_2p_3p_4]$ is a four sided A-patch.

4.4 Summary

In summary, we give an outline of the algorithm. The last step can be incorporated into one, depending on the implementation.

Algorithm 4.3 *Construction of C^2 A-patch surface*

INPUT: a triangulated polyhedron $\mathcal{P}$.

- Compute the normal at each vertex of $\mathcal{P}$.
- Construct C^1 cubic simplicial hull and C^1 A-patches
- Degree raising the cubic simplicial hull to quintic. Used as default values.
- Estimate C^2 data (Hessian matrix) at each vertex. interpolate them with the polynomials.
- Construct piecewise C^2 quintic polynomials.
- Set functional zero contouring conditions.

OUTPUT: C^2 A-patch surface.

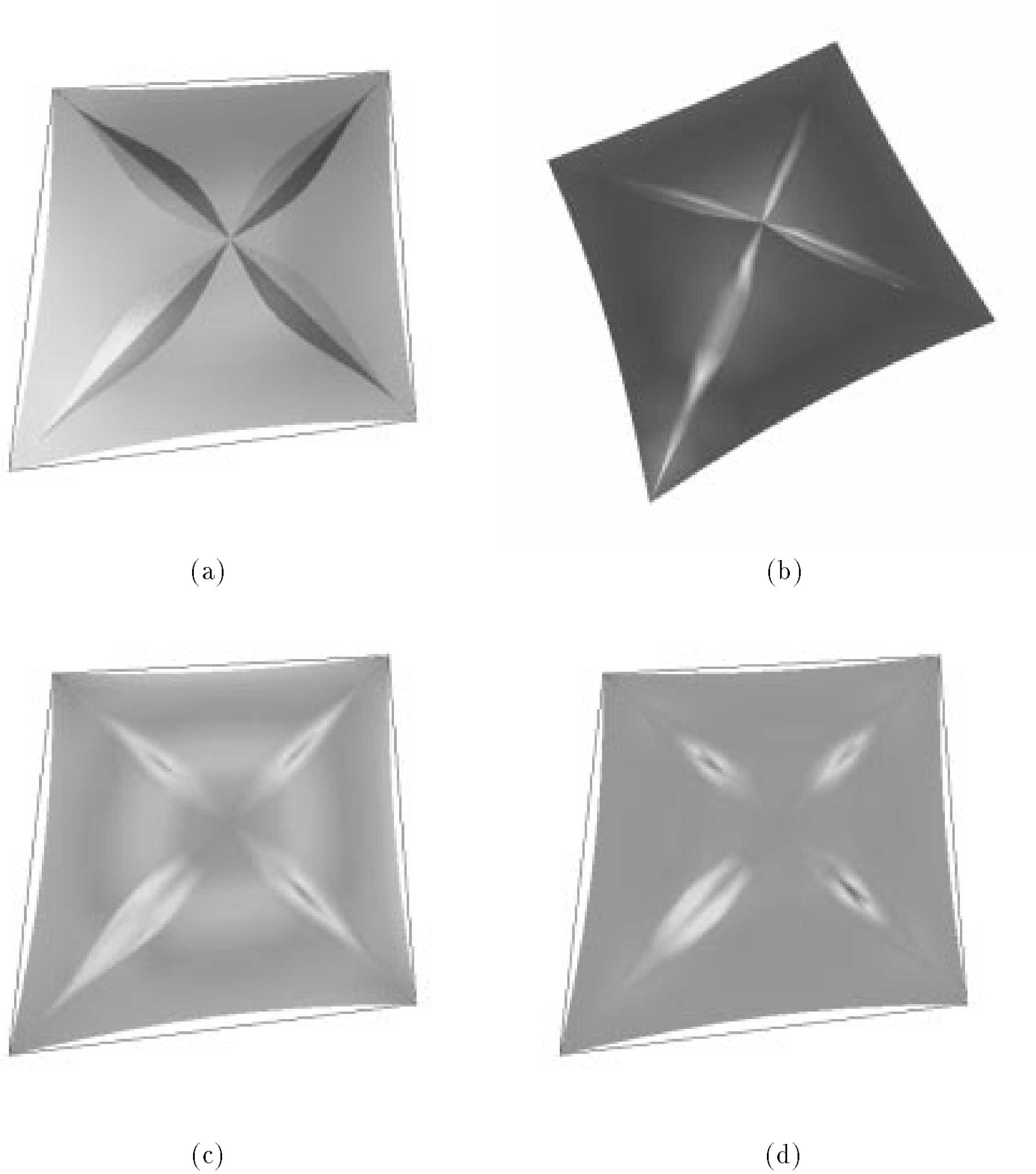
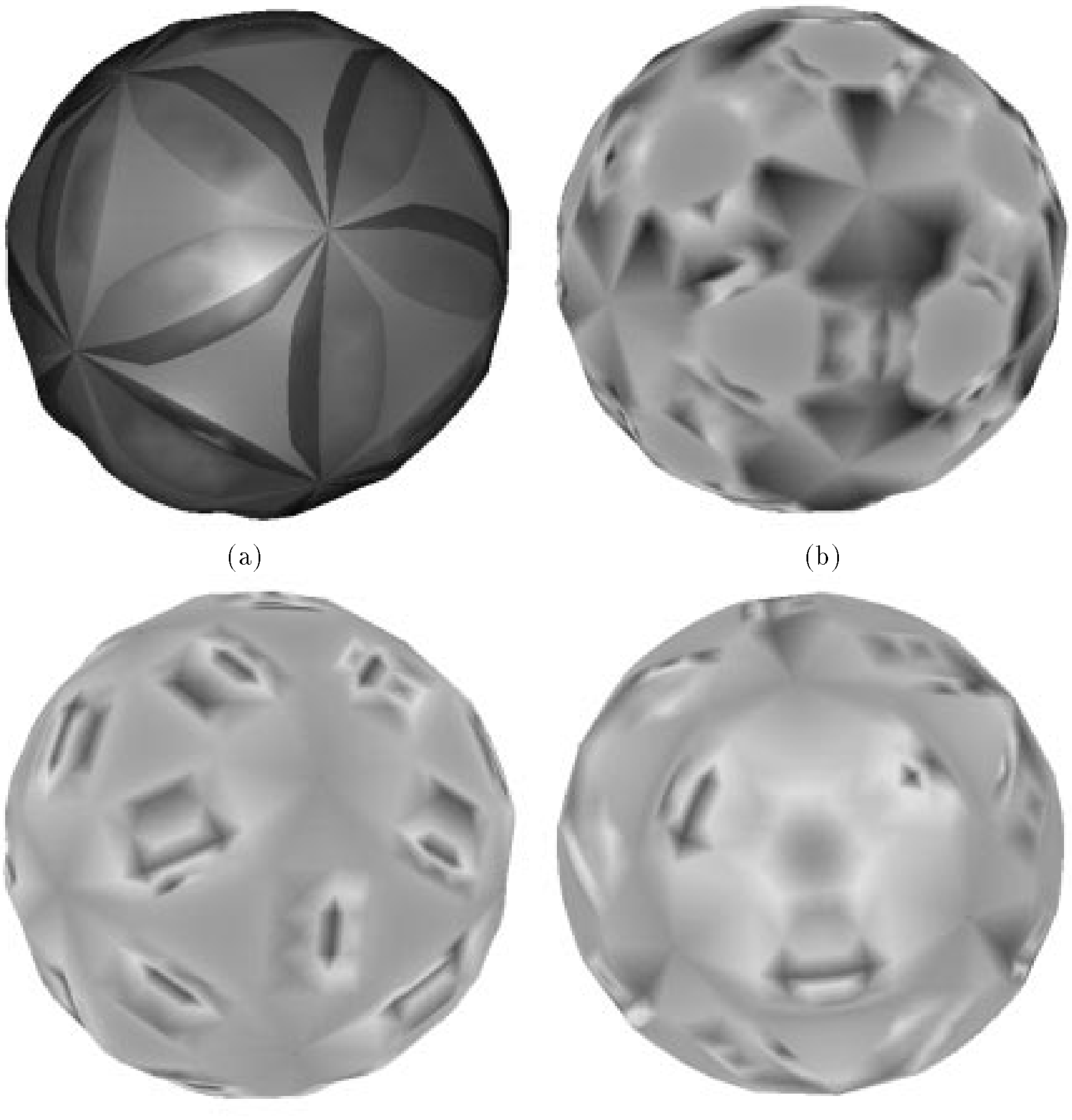


Figure 5.9: A simple C^2 example. (a) Drawn in different shades to show the piecewise structure. (b) C^1: Showing Gaussian curvature. (c) C^2: Showing Mean curvature. (d) C^2: Showing Gaussian curvature.



(a) (b) (c) (d)

Figure 5.10: A C^2 smooth icosahedron. (a) Drawn in different shades to show the piecewise structure. (b) Gaussian curvature. (c) Mean curvature. (d) Mean curvature. The surface is modified locally around the patch facing up.

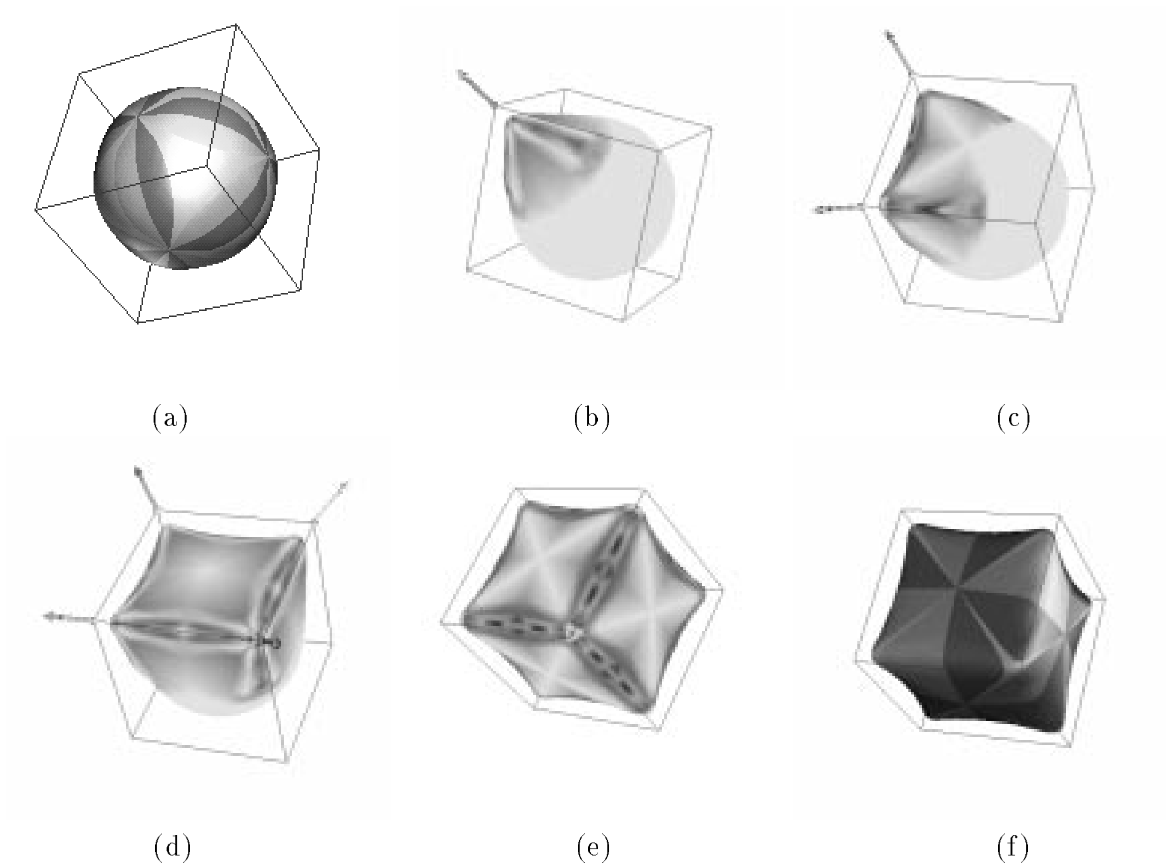


Figure 5.11: Interactive control of C^2 A-patches. Starting from a sphere, the quintic piecewise surface is deforming toward a cube. (a) Sphere. (b) The surface is dragged toward a vertex of the cube. Mean curvature map. (c) Toward an edge of the cube. Gaussian curvature. (d) Toward an face of the cube. Mean curvature. (e) Toward the cube. Gaussian curvature. (f) Toward the cube. Shaded in pattern to show the piecewise structure.

5 Implementation

We have presented algorithms for interpolating a three dimensional polyhedron with C^2 quintic A-patches. These algorithms have been implemented in the SplineX and Shilp toolkits of Shastra, a distributed and collaborative geometric design environment [AB93]. Shilp is an X-11 based, interactive solid modeling system and is used to create a simplicial (face triangulated) polyhedral model of the desired shape. This model could also be the triangulation of an arbitrary surface in three dimensions. This triangulation is C^1 smoothed by a client/server call to a SplineX computation using inter process communication. SplineX is a an X-11 based, interactive surface modeling toolkit for arbitrary algebraic surfaces (implicit or parametric) in BB form. It allows for the creation of simplex chains (as for example the simplicial hull of the triangulation) and the interactive change of control points and weights of the A-patches for shape control. SplineX also has the ability to distribute its rendering tasks (for the display of the individual A-patches) on a network of workstations, to achieve maximal display parallelism.

Figure 5.9 shows a simple example, quintic A-patches based on 4 triangles sharing a vertex. Figure 5.10 shows quintic A-patches built on top of a icosahedron. Note that the sizes of edge patches in pairs are not even, as an convention simplicial hull based on an icosahedron, with equal sized edge patch pair, would cause case 1 degeneracy(See Appendix B). Figure 5.11 shows interactive control of C^2 quintic A-patches. Starting from a sphere, we drag the surface towards the vertices of a cube by changing the weights a_{0113}, a_{1013} and a_{1103} of the quintic face simplexes. Note that the simplicial hull in Figure 5.11 suffers case 2 degeneracy(See Appendix B). Hence we keep weight a_{1112} as is, but instead change a_{0113}, a_{1013} and a_{1103}.

6 Conclusion

We give piecewise C^2 quintic A-patch scheme to C^2 interpolate or approximate each vertex of a given polyhedron. However, as a first step toward C^2 algebraic splines in BB form, the scheme still has many problems. Compared to the C^1 cubic A-patch scheme [BCX95a], this scheme is more complicated in the following aspects. First, we have to solve larger linear system to achieve C^2 continuity. Secondly, it is far more difficult to guarantee that the surface is fully connected, free of singularity and unwanted branches, as in general, the "one sign change" principle is not practical. Thirdly, there are quite many degenerate cases, where the linear systems are singular and we have modified the simplicial hull to remove the degeneracies.

However, it seems to us there is not much room that we can play with quintic surface for C^2 continuity, under the same simplicial hull structure. Further improvements can be made by studying more about the degenerated cases, using higher degree surfaces, or different simplicial hull structures. For example, if we use degree 7 patches, which interpolates some given or pre-calculated C^3 data at each vertex, and C^2 elsewhere, we are able to break up those big systems around the vertices into smaller ones around the edge. Namely, the systems involve no more than 8 simplexes, same level as in the C^1 cubic case.

Besides all these problem, the scheme inherits some common problem of surface fitting, as shown in Figure 5.10, although C^2 smooth, the surface looks bumpy. An optimization scheme is hence needed to construct fair surface with respect to this particular problem.

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A Determining Number 5 and 6 Weights of Quintic

It follows from (2.10) and (2.11) that

$$b_{1211}^{(i)} = \beta_1^{(i)} a_{1211}^{(i)} + \beta_2^{(i)} a_{0311}^{(i)} + \beta_3^{(i)} a_{0221}^{(i)} + \beta_4^{(i)} a_{0212}^{(i)} \tag{A.42}$$

$$\begin{aligned} b_{2210}^{(i)} &= \beta_1^{(i)}\beta_1^{(i)} a_{2210}^{(i)} + 2\beta_1^{(i)}\beta_2^{(i)} a_{1310}^{(i)} + 2\beta_1^{(i)}\beta_3^{(i)} a_{1220}^{(i)} + 2\beta_1^{(i)}\beta_4^{(i)} a_{1211}^{(i)} \\ &+ \beta_2^{(i)}\beta_2^{(i)} a_{0410}^{(i)} + 2\beta_2^{(i)}\beta_3^{(i)} a_{0320}^{(i)} + 2\beta_2^{(i)}\beta_4^{(i)} a_{0311}^{(i)} \\ &+ \beta_3^{(i)}\beta_3^{(i)} a_{0230}^{(i)} + 2\beta_3^{(i)}\beta_4^{(i)} a_{0221}^{(i)} + \beta_4^{(i)}\beta_4^{(i)} a_{0212}^{(i)} \end{aligned} \tag{A.43}$$

for $i = 1, 2$. (A.43) can be written briefly as

$$b_{2210}^{(i)} = 2\beta_1^{(i)}\beta_4^{(i)} a_{1211}^{(i)} + \beta_4^{(i)}\beta_4^{(i)} a_{0212}^{(i)} + \gamma \tag{A.44}$$

where γ is the known terms in (A.43). Since

$$b_{2210}^{(1)} = \mu_1 b_{1211}^{(1)} + \mu_2 b_{1211}^{(2)} \tag{A.45}$$

$$\mu_1^2 b_{0212}^{(1)} - \mu_1 b_{1211}^{(1)} = \mu_2^2 b_{0212}^{(2)} - \mu_2 b_{1211}^{(2)} \tag{A.46}$$

then by substituting (A.42) into (A.45) and (A.46) and then eliminating $b_{2210}^{(i)}$ from (A.44) and (A.45) we get three equations related to four unknowns which could be written as:

$$\begin{bmatrix} \beta_4^{(1)} - \mu_1 & -\mu_2 \\ -\mu_1 & \beta_4^{(2)} - \mu_2 \end{bmatrix} \begin{bmatrix} \beta_4^{(1)} & 0 \\ 0 & \beta_4^{(2)} \end{bmatrix} \begin{bmatrix} a_{0212}^{(1)} \\ a_{0212}^{(2)} \end{bmatrix} = - \begin{bmatrix} 2\beta_4^{(1)} - \mu_1 & -\mu_2 \\ -\mu_1 & 2\beta_4^{(2)} - \mu_2 \end{bmatrix} \begin{bmatrix} a_{1211}^{(1)} \\ a_{1211}^{(2)} \end{bmatrix} + \cdots \tag{A.47}$$

$$\left[-\mu_1(\beta_4^{(1)} - \mu_1) \;\; \mu_2(\beta_4^{(2)} - \mu_2)\right] \begin{bmatrix} a_{0212}^{(1)} \\ a_{0212}^{(2)} \end{bmatrix} - \left[\mu_1\beta_1^{(1)}, -\mu_2\beta_1^{(2)}\right] \begin{bmatrix} a_{1211}^{(1)} \\ a_{1211}^{(2)} \end{bmatrix} = \cdots \tag{A.48}$$

where $\cdots$ are known terms. Since the coefficient matrix of (A.47) is nonsingular, by solving $[a_{0212}^{(1)}\ a_{0212}^{(2)}]^T$ from (A.47) and then substituting it into (A.48), we get one equation relating to the unknowns $a_{1211}^{(1)}$, $a_{1211}^{(2)}$. Let the equation be in the form

$$\phi a_{1211}^{(1)} + \psi a_{1211}^{(2)} = \omega \tag{A.49}$$

Therefore, these unknowns form a closed chain around the vertex p_2 in one side of the tangent plane at p_2.

Let

$$A = \begin{bmatrix} \phi_1 & \psi_1 & & \\ & \phi_2 & \psi_2 & \\ & & \ddots & \\ \psi_k & & & \phi_k \end{bmatrix} \tag{A.50}$$

$$\overline{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \ddots \\ \omega_k \end{bmatrix} \tag{A.51}$$

We observe that system

$$Ax = \overline{\omega} \tag{A.52}$$

is of rank $k-1$. In other words, it has infinite number of solutions. We have not come up with a proof of this property yet. Assuming this observation is true, we choose the solution that least-squared approximate the default values.

B Determining Number 9 Weights of the Quintic Scheme

For $i = 1, 2$,

$$b^{(i)}_{1202} = \beta^{(i)}_1 a^{(i)}_{1202} + \beta^{(i)}_2 a^{(i)}_{0302} + \beta^{(i)}_3 a^{(i)}_{0212} + \beta^{(i)}_4 a^{(i)}_{0203} \tag{B.53}$$

$$\begin{aligned} b^{(i)}_{2201} &= \beta^{(i)}_1\beta^{(i)}_1 a^{(i)}_{2201} + 2\beta^{(i)}_1\beta^{(i)}_2 a^{(i)}_{1301} + 2\beta^{(i)}_1\beta^{(i)}_3 a^{(i)}_{1211} + 2\beta^{(i)}_1\beta^{(i)}_4 a^{(i)}_{1202} \\ &+ \beta^{(i)}_2\beta^{(i)}_2 a^{(i)}_{0401} + 2\beta^{(i)}_2\beta^{(i)}_3 a^{(i)}_{0311} + 2\beta^{(i)}_2\beta^{(i)}_4 a^{(i)}_{0302} \\ &+ \beta^{(i)}_3\beta^{(i)}_3 a^{(i)}_{0221} + 2\beta^{(i)}_3\beta^{(i)}_4 a^{(i)}_{0212} + \beta^{(i)}_4\beta^{(i)}_4 a^{(i)}_{0203} \end{aligned} \tag{B.54}$$

and

$$b^{(1)}_{3200} = \mu_1 b^{(1)}_{2201} + \mu_2 b^{(2)}_{2201} \tag{B.55}$$

$$\mu_1^2 b^{(1)}_{1202} - \mu_1 b^{(1)}_{2201} = \mu_2^2 b^{(2)}_{1202} - \mu_2 b^{(2)}_{2201} \tag{B.56}$$

Substitute (B.53) and (B.54) into (B.56), we have

$$\mu_1\beta^{(1)}_4(\mu_1 - \beta^{(1)}_4)b^{(1)}_{0203} - \mu_2\beta^{(2)}_4(\mu_2 - \beta^{(2)}_4)b^{(2)}_{0203} = \cdots$$

This is a system of the form

$$A = \begin{bmatrix} \phi_1 & \psi_1 & & \\ & \phi_2 & \psi_2 & \\ & & \ddots & \\ \psi_k & & & \phi_k \end{bmatrix}$$

whose determinant is $\prod_{i=1}^{k+r}\phi_i - (-1)^{k+r}\prod_{i=1}^{k+r}\psi_i$. This matrix is nonsingular in general if the points given are in the general position. Hence the system can be solved.

However, if the surrounding tetrahedra at the same side of p_2 are not closed, the matrix A is in the form of

$$A = \begin{bmatrix} \phi_1 & \psi_1 & & \\ & \ddots & \ddots & \\ & & \phi_k & \psi_k \end{bmatrix}$$

By choosing one unknown, say the l-th to be a free parameter, A can be written as $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ with

$$A_1 = \begin{bmatrix} \phi_1 & \psi_1 & & \\ & \ddots & \ddots & \\ & & \phi_{l-1} & \psi_{l-1} \\ & & & \phi_l \end{bmatrix}, \quad A_2 = \begin{bmatrix} \psi_{l+1} & & & \\ \phi_{l+2} & \psi_{l+2} & & \\ & \ddots & \ddots & \\ & & \phi_k & \psi_k \end{bmatrix}$$

Hence the system of equations breakup into two smaller sub-systems. Each of them can be solved separately.

We identify two kinds of degeneracies as follows.

(1) $\beta_4^{(1)} = \beta_4^{(2)}$, but not equal to zero.
μ_1 and μ_2 need to be set to some values other than 0.5 so that A is not singular.

(2) $\beta_4^{(1)}$ and $\beta_4^{(2)}$ are all zero.
In this case, changing μ_1, μ_2 would not solve the problem. We perturbed the tops of the face simplexes to avoid singularity in A.

C Enforcing positivity of quartic and cubic BB polynomials

C.1 Enforcing positivity of a cubic polynomial

For a cubic BB polynomial

$$F3(\alpha) = b_{30}B_{30}^3(\alpha) + b_{21}B_{21}^3(\alpha) + b_{12}B_{12}^3(\alpha) + b_{03}B_{03}^3(\alpha) \tag{C.57}$$

where $b_{30} > 0$, b_{21} are given, set b_{12} and b_{03} so that $F3(\alpha) > 0$ in $[0,1]$.

We observe that, if $F3(\alpha) > 0$, then there must exist a subdivision of the BB polynomial where the weights of each piece are all positive. We subdivide the univariate cubic polynomial $F3(\alpha)$ into two pieces,

$$F3^{(1)} = b_{30}^{(1)}B_{30}^3(\alpha) + b_{21}^{(1)}B_{21}^3(\alpha) + b_{12}^{(1)}B_{12}^3(\alpha) + b_{03}^{(1)}B_{03}^3(\alpha) \tag{C.58}$$

$$F3^{(2)} = b_{30}^{(2)}B_{30}^3(\alpha) + b_{21}^{(2)}B_{21}^3(\alpha) + b_{12}^{(2)}B_{12}^3(\alpha) + b_{03}^{(2)}B_{03}^3(\alpha) \tag{C.59}$$

upon $\alpha = t$ (See figure 4.4). We now just need to choose $t = t_0 < \frac{b_{30}}{b_{30} - b_{21}}$ so that

$$b_{21}^{(1)} = (1 - t_0)b_{30} + t_0 b_{21} > 0 \tag{C.60}$$

The other weights are positive function of b_{12} and b_{03} and hence can be set to be positive. In particular,

$$b_{12}^{(1)} = (1 - t_0)b_{21}^{(1)} + t_0((1 - t_0)b_{21} + tb_{12}) \tag{C.61}$$

$$> t_0((1 - t_0)b_{21} + t_0 b_{12}) \tag{C.62}$$

Hence if

$$\begin{aligned} (1 - t_0)b_{21} + t_0 b_{12} &> 0 \text{ or} \\ b_{12}b_{30} - b_{20}^2 &> 0, \end{aligned} \tag{C.63}$$

$b_{12}^{(1)} > 0$. It is trivial to verify that if furthermore $b_{03} > 0$, all other control points are positive.

C.2 Enforcing positivity of a quartic polynomial

For a quartic BB polynomial,

$$F4(\alpha) = b_{40}B_{40}^4(\alpha) + b_{31}B_{31}^4(\alpha) + b_{22}B_{22}^4(\alpha) + b_{13}B_{13}^4(\alpha) + b_{04}B_{04}^4(\alpha) \tag{C.64}$$

where $b_{40} > \varepsilon$, b_{31} and b_{22} are given, set b_{13} and b_{04} so that $F4(\alpha) > 0$ in $[0,1]$.

We subdivide the univariate quartic polynomial $F4(\alpha)$ into two pieces,

$$F4^{(1)}(\alpha) = b_{40}^{(1)}B_{40}^4 + b_{31}^{(1)}B_{31}^4 + b_{22}^{(1)}B_{22}^4 + b_{13}^{(1)}B_{13}^4 + b_{04}^{(1)}B_{04}^4 \tag{C.65}$$

$$F4^{(2)}(\alpha) = b_{40}^{(2)}B_{40}^4 + b_{31}^{(2)}B_{31}^4 + b_{22}^{(2)}B_{22}^4 + b_{13}^{(2)}B_{13}^4 + b_{04}^{(2)}B_{04}^4 \tag{C.66}$$

upon $\alpha = t$ (see figure 4.6). By choosing t carefully, one makes sure that

$$b_{31}^{(1)} = (1-t)b_{40} + tb_{31} > 0 \tag{C.67}$$

$$b_{22}^{(1)} = (1-t)^2 b_{40} + 2(1-t)tb_{31} + t^2 b_{22} > 0. \tag{C.68}$$

Specifically, Let $r = \frac{1-t}{t}$, inequality (C.68) is equivalent to

$$b_{40}r^2 + 2b_{31}r + b_{22} > 0 \tag{C.69}$$

Solving this inequality in $r > 0$,

$$r > \frac{-b_{31} + \sqrt{b_{31}^2 - b_{40}b_{22}}}{b_{40}} \tag{C.70}$$

Hence by choosing $r = r_0 > \frac{-b_{31}+\sqrt{b_{31}^2 - b_{40}b_{22}}}{b_{40}}$, $b_{22}^{(1)} > 0$. It is easy to verify that so is $b_{31}^{(1)}$.

Then, in order to make $b_{13}^{(1)} > 0$, as

$$b_{13}^{(1)} = (1-t)b_{22}^{(1)} + t(b_{31}B_{20}^2 + b_{22}B_{11}^2 + b_{13}B_{02}^2) > 0 \tag{C.71}$$

or

$$b_{31}B_{20}^2(t_0) + b_{22}B_{11}^2(t_0) + b_{13}B_{02}^2(t_0) > 0 \tag{C.72}$$

where $t_0 = \frac{1}{1+r_0}$, or

$$b_{31}r_0^2 + b_{22}2r_0 + b_{13} > 0 \tag{C.73}$$

or

$$b_{13} > -\frac{-b_{31} + \sqrt{b_{31}^2 - b_{40}b_{22}}}{b_{40}} \left(b_{31}\frac{-b_{31} + \sqrt{b_{31}^2 - b_{40}b_{22}}}{b_{40}} + 2b_{22} \right) \tag{C.74}$$

Furthermore, with $b_{04} > 0$, every other weights are positive.

C.3 Enforcing positivity of a quintic polynomial

For a quintic BB polynomial,

$$F5(\alpha) = b_{50}B^5_{50}(\alpha) + b_{41}B^5_{41}(\alpha) + b_{32}B^5_{32}(\alpha) + b_{23}B^5_{23}(\alpha) + b_{14}B^5_{14}(\alpha) + b_{05}B^4_{05}(\alpha) \quad \text{(C.75)}$$

where $b_{50} > \varepsilon$, b_{41}, b_{32} and b_{23} are given, set b_{14} and b_{05} so that $F5(\alpha) > 0$ in $[0, 1]$.

We subdivide the univariate quintic polynomial $F5(\alpha)$ into two pieces,

$$F5^{(1)}(\alpha) = b^{(1)}_{50}B^5_{50} + b^{(1)}_{41}B^5_{41} + b^{(1)}_{32}B^5_{32} + b^{(1)}_{23}B^5_{23}b^{(1)}_{14}B^5_{14} + b^{(1)}_{05}B^5_{05} \quad \text{(C.76)}$$

$$F5^{(2)}(\alpha) = b^{(2)}_{50}B^5_{50} + b^{(2)}_{41}B^5_{41} + b^{(2)}_{32}B^5_{32} + b^{(2)}_{23}B^5_{23}b^{(2)}_{14}B^5_{14} + b^{(2)}_{05}B^5_{05} \quad \text{(C.77)}$$

upon $\alpha = t$ (see figure 4.5). By choosing t carefully, one makes sure that

$$b^{(1)}_{41} = (1-t)b_{50} + tb_{41} > 0 \quad \text{(C.78)}$$

$$b^{(1)}_{32} = (1-t)^2 b_{50} + 2(1-t)tb_{41} + t^2 b_{32} > 0 \quad \text{(C.79)}$$

$$b^{(1)}_{23} = (1-t)^3 b_{50} + 3(1-t)^2 tb_{41} + 3(1-t)t^2 b_{32} + t^3 b_{23} > 0 \quad \text{(C.80)}$$

Similar to the cubic and quartic case, we solve

$$(1-t)^3 b_{50} + 3(1-t)^2 tb_{41} + 3(1-t)t^2 b_{32} + t^3 b_{23} > 0 \quad \text{(C.81)}$$

in $0 < t < 1$, or its equivalence

$$r^3 b_{50} + 3r^2 b_{41} + 3rb_{32} + b_{23} > 0 \quad \text{(C.82)}$$

with $r = \frac{1-t}{t}$.

Let t_0 be what we choose for t. We next choose b_{14} so that

$$b^{(1)}_{14} = (1-t_0)t^{(1)}_{23} + t_0(b_{41}B^3_{30}(t_0) + b_{32}B^3_{21}(t_0) + b_{23}B^3_{12}(t_0) + b_{14}B^3_{03}(t_0)) \quad \text{(C.83)}$$

$$> t_0(b_{41}B^3_{30}(t_0) + b_{32}B^3_{21}(t_0) + b_{23}B^3_{12}(t_0) + b_{14}B^3_{03}(t_0)) > 0 \quad \text{(C.84)}$$

or

$$b_{14} > -(b_{41}r_0^3 + 3b_{32}r_0^2 + 3b_{23}r_0) \quad \text{(C.85)}$$

Setting $b_{05} > 0$ makes every other weights positive.