FINITE REPRESENTATIONS OF REAL PARAMETRIC CURVES AND SURFACES

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Finite representations of real parametric curves and surfaces*

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Abstract

Global parameterizations of parametric algebraic curves or surfaces are defined over infinite parameter domains. In this paper we show how to replace these global real parameterizations with a finite number of alternate bounded parameterizations, each defined over a fixed, bounded part of the real parameter domain space. The new bounded parameterizations together generate all real points of the old one and in particular the points corresponding to infinite parameter values in the old domain. We term such an alternate finite set of bounded parameterizations a ***finite representation*** of a real parametric curve or surface. Two solutions are presented for real parametric varieties of arbitrary dimension n. In the first method, a real parametric variety of dimension n is finitely represented in a piecewise fashion by 2^n bounded parameterizations with individual pieces meeting with C^∞ continuity; each bounded parameterization is a map from a unit simplex of the real parameter domain space. In the second method, only a single bounded parameterization is used; it is a map from the unit hypersphere centered at the origin of the real parameter domain space. Both methods start with an arbitrary real parameterization of a real parametric variety and apply projective domain transformations of different types to yield the new bounded parameterizations. Both these methods are implementable in a straightforward fashion. Applications of these results include displaying entire real parametric curves and surfaces, computing *normal* parameterizations of curves and surfaces (settling an open problem for quadric surfaces), and exactly representing an entire real parametric curve or surface in piecewise Bernstein-Bezier form.
Keywords: real parametric curves and surfaces, varieties, computer graphics, projective domain transformations.

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1 Introduction

Algebraic curves and surfaces are commonly used in geometric modeling. *Parametric* curves and surfaces are those that can be represented using rational parametric equations, and are particularly important. In geometric modeling applications, the parametric equations are restricted to some bounded portion of the domain, yielding a segment of a curve or a patch of a surface. However, the algebraic curve or surface is an image of the entire infinite parameter domain. Attempting to map the entire curve or surface using very large regions of the parameter domain is not a solution because some finite points may be images of infinite parameter values.

Thus a natural question arises: can one cover an entire curve or surface, using only a finite number of bounded regions of the parameter domain ? This is indeed possible, and two methods are described in this paper.

In the first method, a given rational parameterization is replaced by several bounded parameterizations that together generate all the points that the original one did, including points that correspond to infinite parameter values. Projective linear domain transformations (reparameterizations) are applied that map the unit simplex of each parameter domain onto an entire octant of the original parameter domain space in turn. This approach for the special case of real parametric curves and surfaces in the Bernstein-Bezier form, is similar to the technique called homogeneous sampling [9] used to sample finite and infinite domain points of a parameterization equally. One application of our work is displaying entire real parametric curves and surfaces. Another application is the first step towards representing an entire real parametric curve (surface) by a collection of curves (surfaces) in Bernstein-Bezier form, each with positive weights. This possibility is raised in [17].

In the second approach, it is shown that a *single* projective reparameterization suffices to map all finite and infinite parameter values of the old parameterization, using only finite values of the new parameter domain. In this case the reparameterization is quadratic and the region of the domain space that suffices is the unit hypersphere of the new domain space. Because of the higher degree and the non-linear boundary of the domain region, this approach is less practical, but it can be used to compute *normal parameterizations* of curves and surfaces – that is, parameterizations that map all points of the curve or surface, without "missing" any. This issue was discussed in [11], where normal parameterizations for conic curves and some quadric surfaces were given. The problem of computing normal parameterizations for three important quadric surfaces was left open, and we shall give the solutions here.

Since the results generalize to higher dimensions, our discussion will be in terms of real parametric varieties of any dimension. The problem can be stated as follows. Given a real parameterization of a parametric variety, we would like to compute an alternate set of bounded parameterizations that together generate all the real points of the variety: those that correspond to finite parameter values, and those that correspond to infinite parameter values (in the original parameterization).

This paper is organized as follows. In the next section some preliminary definitions and terminology are given, and the issue of "missing points" is discussed in some detail. In section 3, we show how to finitely represent a real parametric variety of dimension n using 2^n pieces. In section 4 it is shown that a single reparameterization is sufficient, and in section 5 we make some concluding remarks and indicate directions for future work.

2 Finite parametric representations

The set of solutions of a set of polynomial equations with real coefficients in m variables forms a real algebraic set in $\mathcal{R}^m$, where $\mathcal{R}$ is the field of real numbers. A real algebraic set that cannot be properly represented as the union of two real algebraic sets is called a real *variety*. A *parametric* variety is one whose points can be given as the image of a map over some domain space. We restrict our attention to maps defined by rational functions.

Let the points of a variety V of dimension n in R^m $(n < m)$ be given by a rational-function map in n parameters:

$$V(\mathbf{s}) = \begin{pmatrix} x_1(s_1, \ldots, s_n) \\ \vdots \\ x_m(s_1, \ldots, s_n) \end{pmatrix}, \qquad s_i \in (-\infty, +\infty)$$

The rational functions $x_i(s_1, \ldots, s_n)$ constitute a *parameterization* of the variety and are assumed to have a common denominator. Methods exist for computing rational parameterizations of various classes of varieties [1, 2, 3, 4, 13, 14, 16, 21]. All these algorithms generate parameterizations of curves and surfaces over infinite domains.

We view the map as one from the real projective space of n dimensions to the real affine space of m dimensions, i.e. $V(\mathbf{s}) : \mathcal{RP}^n \to \mathcal{R}^m$. By doing so, we allow each parameter s_i to take on any value in $\mathcal{R}$ as well as the value ∞. It is often the case that a finite point (one in $\mathcal{R}^m$) of the variety given by rational functions is mapped by an infinite parameter value in $\mathcal{RP}^n$.

For example, a 1-dimensional real variety in $\mathcal{R}^2$ is defined by the bivariate polynomial equation $x^2 + y^2 - 1 = 0$. The points are in the image of the univariate rational functions

$$x(s) = \frac{2s}{s^2+1}, \qquad y(s) = \frac{s^2-1}{s^2+1}, \qquad s \in (-\infty, +\infty) \tag{1}$$

Notice that the point $(0, 1) \in \mathcal{R}^2$ of the variety is the image of the parameter value $s = \infty \in \mathcal{RP}$.

2.1 Missing points of parameterizations

There are two categories of potential *missing points* of a real parametric representation of a variety.

First, a parametric variety may have finite points that correspond to infinite parameter values. The methods in [11, 23], though computation intensive, exhibit a way to prove whether or not a given parameterization has such missing points and to compute them. Two different solutions are provided in this paper for dealing with missing points and generating new parameterizations which do not have missing points.

Examples. A simple case is the unit circle, whose parameterization (1) and missing point were given earlier. Figure 1 shows two surfaces, an ellipsoid and a Steiner surface. A point is missing on the ellipsoid, and a curve from the Steiner surface. The parameterizations of the ellipsoid and the Steiner surface are given in Table 1. For each parameterization we show the image of a rectangular domain region centered at the origin. All such images show a "hole" or gap (the clover-leaf on the ellipsoid is a hole).

Increasing the area of the domain region will shrink the gap but never close it. Furthermore, if the domain region is discretized uniformly to generate a piecewise-linear mesh approximating the surface, the pieces tend to be large away from the gap, but small and dense near the gap; curvature-sensitive approximation techniques are necessary.

Second, parametric varieties of dimension greater than 1 can have *base points*, which are points in the parameter domain at which all numerators and (common) denominator of a parameterization vanish. For surfaces it has been shown that a domain base point corresponds to an entire curve on the surface [15]. For example, consider the variety defined by $x^2 + y^2 - z^2 - 1 = 0$. A parameterization for it is

$$(x(u,v), y(u,v), z(u,v)) = \left(\frac{u^2 - v^2 + 1}{u^2 + v^2 - 1}, \quad \frac{2uv}{u^2 + v^2 - 1}, \quad \frac{2u}{u^2 + v^2 - 1} \right)$$

Figure 1: Missing points due to infinite parameter values

which has the base points $(u,v) = (0, \pm 1)$; it can be shown that these points map onto the lines $(x(s), y(s), z(s)) = (-1, s, s)$ and $(x(t), y(t), z(t)) = (-1, t, -t)$ on the surface. Essentially, approaching the domain base point from two different directions leads to different surface points, in the limit.

Domain base points cause "pseudo" missing points on the variety that are the image of finite parameter values. These points on the variety are only missing in that the rational map itself is ill-defined, when specialized to the domain base points. We don't present a reparameterization solution to this problem but leave it open for future research.

One way for surfaces is as follows: besides reparameterization, one may augment the existing parameterization with parameterizations of the image points corresponding to base points, as suggested in [8]. Such space curves on the surface are called *seam curves*, and are known to be rational. Algorithms for computing rational-map parameterizations of these curves are given in [18], but they are not practical at this time. A more practical approach might be to numerically approximate the seam curves.

2.2 Problem statement

We wish to replace a parameterization over an infinite parameter domain with a finite number of parameterizations, each over a fixed, bounded parameter domain. Suppose we are given a parameterization $V(\mathrm{s}) : \mathcal{RP}^n \to \mathcal{R}^m$ of a variety. We wish to compute maps $Q_1, \ldots, Q_k$, with $Q_i : \mathcal{R}^n \to \mathcal{R}^m$, such that $\cup_{i=1}^k Q_i(\mathcal{R}^n) = V(\mathcal{RP}^n)$. That is, the new maps restricted to finite values together yield the same set of points that the given one does, even though the latter maps both finite and infinite domain values. To derive a *finite representation* we also find a bounded region $D \subset \mathcal{R}^n$ to which the Q_i can be be restricted, i.e., $\cup_{i=1}^k Q_i(D) = V(\mathcal{RP}^n)$.

2.3 Main results

Given one parameterization of V, it is possible to compute a finite representation of it, and we show two ways below.

In the first way, an affine variety of dimension n is finitely represented using 2^n parameterizations, with the bounded domain D being the unit simplex in $\mathcal{R}^n$. In the second way, only one parameterization suffices, but its degree is twice that of the original, and the bounded domain D is the unit hypersphere in $\mathcal{R}^n$. The second approach can be used to compute parameterizations of real parametric varieties that are free of *missing points.*

3 Piecewise finite representation

Suppose we are given a real parametric variety V of dimension n and a parameterization $V(\mathbf{s})$ for it. We compute 2^n parameterizations, each restricted to the unit simplex of the parameter domain $\mathcal{R}^n$, that together give all the points that $V(\mathbf{s})$ did for $\mathbf{s} \in \mathcal{RP}^n$.

We use linear projective domain transformations (reparameterizations) to map, in turn, the unit simplex D of the new parameter domain space onto an entire octant of the original parameter domain space. The reparameterizations are specified in affine fractional form for convenience, but in practice they would be applied by homogenizing a parameterization and then substituting polynomials.

THEOREM 1 *Consider a real parametric variety in $\mathcal{R}^m$ of dimension n, $n < m$, which is parameterized by the equations*

$$V(\mathbf{s}) = \begin{pmatrix} x_1(s_1, \ldots, s_n) \\ \vdots \\ x_m(s_1, \ldots, s_n) \end{pmatrix}, \qquad s_i \in (-\infty, +\infty)$$

Let the 2^n octant cells in the parameter domain $\mathcal{R}^n$ be labelled by the tuples $< \sigma_1, \ldots \sigma_n >$ with $\sigma_i \in \{-1, 1\}$. Then the 2^n projective reparameterizations $V(\mathbf{t}_{<\sigma_1,\ldots,\sigma_n>})$ given by

$$s_i = \sigma_i \frac{t_i}{1 - t_1 - t_2 - \ldots - t_n}, \qquad i = 1, \ldots, n \tag{2}$$

together map all the points of the variety $V(\mathbf{s}), s_i \in (-\infty, +\infty)$, using only parameter values satisfying $t_i \geq 0$ and $t_1 + t_2 + \ldots + t_n \leq 1$.

PROOF. We must show that every point in the old domain $\mathcal{RP}^n$ is the image of some point in the new domain $\mathcal{R}^n$. In particular, we show that the hyperplane $t_1 + \ldots + t_n = 1$ bordering the unit simplex in $\mathcal{R}^n$ maps onto the hyperplane at infinity in $\mathcal{RP}^n$, and the rest of the points of the unit simplex are mapped onto a particular octant of the original domain space, depending on the signs of the σ_i.

Let $\mathbf{s} = (cs_1, \ldots, cs_n, cs_{n+1}) \in \mathcal{RP}^n$, where $c \in \mathcal{R}$ is a non-zero constant of proportionality and $s_{n+1} = 0$ is the equation of the hyperplane at infinity in $\mathcal{RP}^n$. Let $\mathbf{t} = (t_1, \ldots, t_n) \in \mathcal{R}^n$. Since (2) is a map from $\mathcal{R}^n \rightarrow \mathcal{RP}^n$, the following relationship holds between the s_i and t_j, under one of the 2^n transformations $< \sigma_1, \ldots, \sigma_n >$:

$$\begin{aligned} cs_1 &= \sigma_1 t_1 \\ &\vdots \\ cs_n &= \sigma_n t_n \\ cs_{n+1} &= 1 - (t_1 + \ldots + t_n) \end{aligned}$$

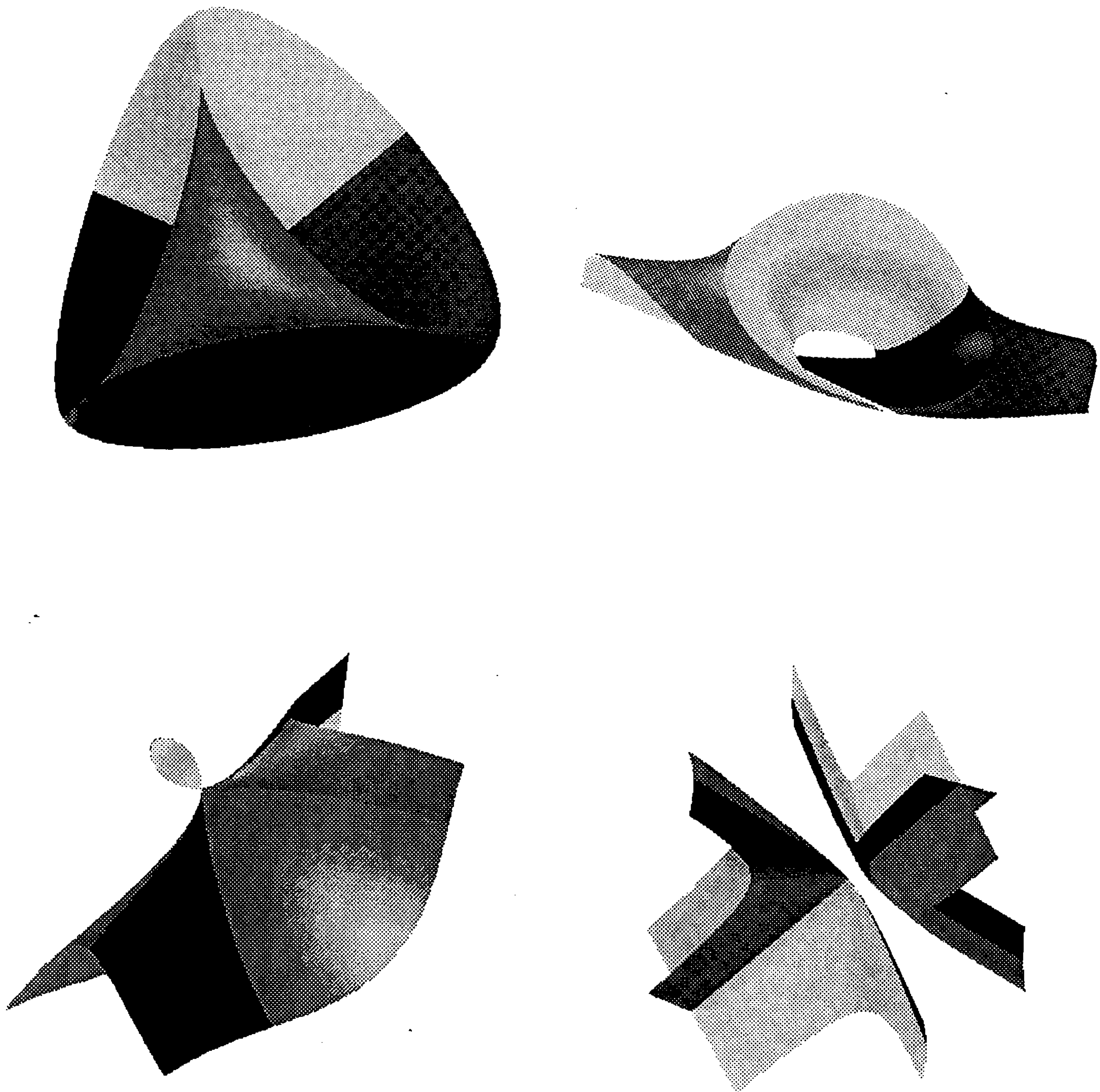

Figure 2: Piecewise finite representations

Let $\operatorname{sign}(\alpha), \alpha \in \mathcal{R}$ be -1 or $+1$ according to whether $\alpha < 0$ or $\alpha \geq 0$, respectively.

First we show that every $\mathbf{s} \in \mathcal{RP}^n$ on the hyperplane at infinity is the image of some point $\mathbf{t} = (t_1, \ldots, t_n) \in \mathcal{R}^n$ under one of the transformations, and additionally that $t_i \geq 0$ and $t_1 + \ldots + t_n = 1$.

Since $\mathbf{s}$ is on the hyperplane at infinity, $s_{n+1} = 0$, and hence

$$\begin{aligned} cs_i &= \sigma_i t_i & i = 1, \ldots, n \\ 0 &= 1 - (t_1 + \ldots + t_n) \end{aligned}$$

Then a solution $(t_1, \ldots, t_n)$ is derived by setting

$$\begin{aligned} \sigma_i &= \operatorname{sign}(s_i) \\ c &= \frac{1}{\sum_{i=1}^n \sigma_i s_i} \\ t_i &= \frac{\sigma_i s_i}{\sum_{i=1}^n \sigma_i s_i} \end{aligned}$$

Noting that $\sigma_i s_i \geq 0$ and not all of the $s_i, i = 1, \ldots n$ can be zero, it follows that $t_i \geq 0$ and $\sum_{i=1}^n t_i = 1$.

Second, let $\mathbf{s} \in \mathcal{R}^n \subset \mathcal{RP}^n$, i.e. $s_{n+1} \neq 0$. We show that $\mathbf{s}$ is the image of some $\mathbf{t} \in \mathcal{R}^n$ under one of the transformations, and additionally $\mathbf{t}$ lies in the unit simplex of $\mathcal{R}^n$.

We can set $s_{n+1} = 1$ w.l.o.g. and the following system of equations for the t_i is derived:

$$\begin{aligned} cs_i &= \sigma_i t_i & i = 1, \ldots, n \\ c &= 1 - (t_1 + \ldots + t_n) \end{aligned}$$

We can solve this linear system by setting

$$\begin{aligned} \sigma_i &= \operatorname{sign}(s_i) \\ c &= \frac{1}{1 + \sum_{i=1}^n \sigma_i s_i} \\ t_i &= \frac{\sigma_i s_i}{1 + \sum_{i=1}^n \sigma_i s_i} \end{aligned}$$

and since $\sigma_i s_i \geq 0$ it follows that $t_i \geq 0$ and $t_1 + \ldots + t_n < 1$, hence this point $\mathbf{t}$ is in the unit simplex in $\mathcal{R}^n$, but not on the hyperplane $t_1 + \ldots + t_n = 1$.

We have thus proved that all of $\mathcal{RP}^n$ is mapped by the transformations (2), restricting each to the unit simplex of $\mathcal{R}^n$. □

COROLLARY 1 *Rational curves given by a parameterization* $C(s) = (x_1(s), \ldots, x_m(s))^T$, $s \in (-\infty, +\infty)$ *can be finitely represented by* $C(\frac{t}{1-t}), C(\frac{-t}{1-t})$, *using only* $0 \leq t \leq 1$.

COROLLARY 2 *Rational surfaces given by a parameterization* $S(s_1, s_2) = (x_1(s_1, s_2), \ldots, x_m(s_1, s_2))^T$, $s_1, s_2 \in (-\infty, +\infty)$ *can be finitely represented by*

$$\begin{array}{ll} S(\frac{t_1}{1-t_1-t_2}, \frac{t_2}{1-t_1-t_2}) & S(\frac{-t_1}{1-t_1-t_2}, \frac{t_2}{1-t_1-t_2}) \\ S(\frac{-t_1}{1-t_1-t_2}, \frac{-t_2}{1-t_1-t_2}) & S(\frac{t_1}{1-t_1-t_2}, \frac{-t_2}{1-t_1-t_2}) \end{array}$$

using only $t_1, t_2 \geq 0 \wedge t_1 + t_2 \leq 1$.

Examples. Figure 2 shows several piecewise representations of surfaces using corollary 2. Points on a surface that are correspond to a particular quadrant of the parameter domain space $\mathcal{RP}^2$ are given a color unique to that quadrant, and the entire surface is trimmed to some bounding box of $\mathcal{R}^3$. The upper left shows a Steiner quartic variety, whose quadratic parameterization is given in Table 1. In the upper right is an "elbow" cubic variety, whose parameterization is

$$\left(\frac{4t^2+(s^2+6s+4)t-4s-8}{2t^2-4t+s^2+4s+8}, \frac{4t^2+(-s^2-6s-20)t+2s^2+8s+16}{2t^2-4t+s^2+4s+8}, \frac{(2s+6)t^2+(-4s-12)t-s^2-4s}{2t^2-4t+s^2+4s+8}\right)$$

A singular cubic surface appears in the lower left; its parameterization is

$$\left(\frac{t^3+2t+s^3-7s^2+1}{t^3+s^3+1}, \frac{2t^3+2t^2-7s^2t+2s^3+2}{t^3+s^3+1}, \frac{2st-7s^3}{t^3+s^3+1}\right)$$

An arbitrary quartic variety whose parameterization is quadratic is given in the lower right; its parametric equations are

$$\left(\frac{2s}{s^2+t^2-2}, \frac{4ts}{s^2+t^2-2}, \frac{1-s^2-t^2}{s^2+t^2-2}\right)$$

4 Single finite representation

We now consider the problem of finitely representing a real parametric variety using only one parameterization. Put another way, can we find parameterizations of varieties that have no missing points ? In [11], such *normal parameterizations* are given for ellipses and some quadric surfaces. Their method for proving a parameterization normal involves elimination-theoretic computations based on the method of characteristic sets developed for geometric theorem proving [23]. The method is general, but lengthy machine computations are involved and the authors were unable to find normal parameterizations for three important quadrics, namely the ellipsoid, hyperboloid of one sheet, and hyperboloid of two sheets; they pose it as an open problem to either find such normal parameterizations or to prove they don't exist. It is also shown that the lowest-degree normal parameterization of an ellipse is of degree 4, which hints that normal parameterizations for the above quadrics will also need to be of degree 4, at least.

We can achieve the same results simply by using projective reparameterizations to bring points at infinity in the parameter domain to finite distances. Instead of using linear projective reparameterizations, however, we now use quadratic projective reparameterizations to solve this problem.

THEOREM 2 *Consider a real parametric variety of dimension n in $\mathcal{R}^m$, $n < m$, which is parameterized by the equations*

$$V(\mathbf{s}) = \begin{pmatrix} x_1(s_1,\ldots,s_n) \\ \vdots \\ x_m(s_1,\ldots,s_n) \end{pmatrix}, \qquad s_i \in (-\infty,+\infty)$$

The single projective quadratic reparameterization given in fractional affine form as

$$s_i = \frac{t_i}{1-t_1^2-t_2^2-\ldots-t_n^2}, \qquad i=1,\ldots,n \tag{3}$$

yields a finite representation $V(\mathbf{t})$ of the rational variety $V(\mathbf{s})$, restricting $t_1^2+\ldots+t_n^2 \leq 1$.

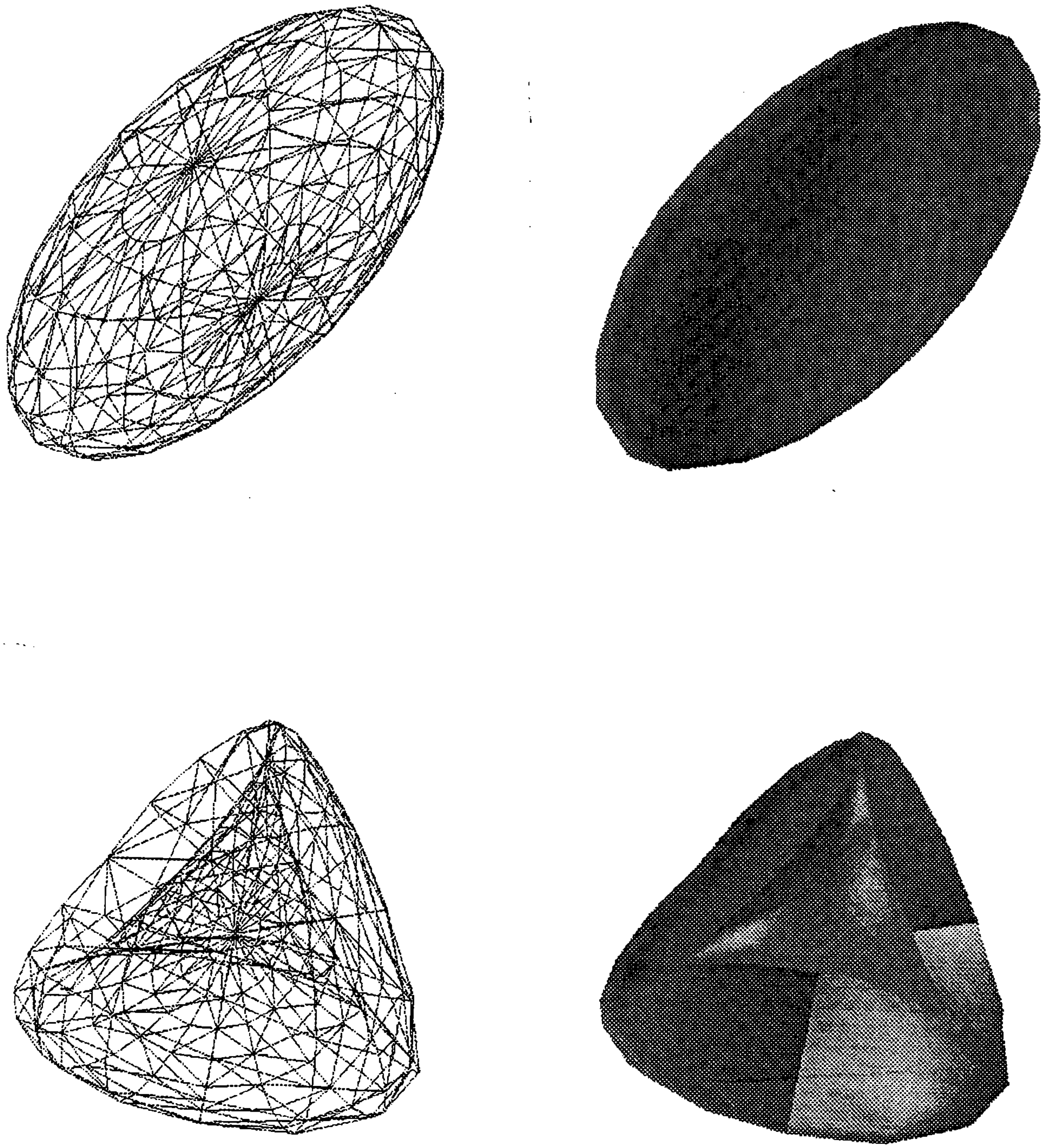

Figure 3: Single finite representations

PROOF. In this case, the proof consists of showing every point in the old domain $\mathcal{RP}^n$ is the image of some point in the new domain $\mathcal{R}^n$, using only the single transformation (3). We will show that the unit hypersphere in the new domain space maps onto the hyperplane at infinity of the old domain space, and every other point in the old parameter domain space is the image of a corresponding point in the new domain, which lies in the interior of the unit hypersphere.

Once again let $\mathbf{s} = (cs_1, \ldots, cs_{n+1}) \in \mathcal{RP}^n$, $c \in \mathcal{R}$, $c \neq 0$ and we fix $s_{n+1} = 0$ as the hyperplane at infinity. Let $\mathbf{t} \in \mathcal{R}^n$. The equations (3) are a map from $\mathcal{R}^n \to \mathcal{RP}^n$:

$$\begin{aligned} cs_i &= t_i \\ &\vdots \\ cs_n &= t_n \\ cs_{n+1} &= 1 - (t_1^2 + \ldots + t_n^2) \end{aligned}$$

First, consider points $\mathbf{s}$ on the hyperplane at infinity, i.e. $s_{n+1} = 0$. Then (3) yields a system of equations

$$\begin{aligned} cs_i &= t_i & i &= 1, \ldots, n \\ 0 &= 1 - (t_1^2 + \ldots + t_n^2) \end{aligned}$$

which has two real solutions, given below:

$$\begin{aligned} c &= \pm \frac{1}{\sqrt{\sum_{i=1}^n s_i^2}} \\ t_i &= cs_i = \pm \frac{s_1}{\sqrt{\sum_{i=1}^n s_i^2}} \end{aligned}$$

For either solution, $t_1^2 + \ldots + t_n^2 = 1$, showing that $\mathbf{t}$ lies on the unit hypersphere in $\mathcal{R}^n$.

Second, consider affine points $\mathbf{s} \in \mathcal{R}^n \subset \mathcal{RP}^n$. We can set $s_{n+1} = 1$, w.l.o.g., and then (3) yields the system of equations

$$\begin{aligned} cs_i &= t_i & i &= 1, \ldots, n \\ c &= 1 - (t_1^2 + \ldots + t_n^2) \end{aligned}$$

This system also has two real solutions, given by

$$\begin{aligned} c &= \frac{-1 \pm \sqrt{1 + 4\sum_{i=1}^n s_i^2}}{2\sum_{i=1}^n s_i^2} \\ t_i &= cs_i \end{aligned}$$

Choosing $c = \frac{-1+\sqrt{1+4\sum_{i=1}^n s_i^2}}{2\sum_{i=1}^n s_i^2}$, some simple algebra shows that $t_1^2 + \ldots t_n^2 < 1$.

Thus if $\mathbf{s}$ is on the hyperplane at infinity, there is a point $\mathbf{t}$ on the unit hypersphere that maps it; otherwise, there is a point $\mathbf{t}$ in the interior of the unit hypersphere that maps it. Only the single map (3) is necessary. $\square$

Given a parameterization of a variety, an application of theorem 2 yields a normal parameterization of the corresponding variety. Thus we can compute normal parameterizations of the ellipsoid and one- and two-sheeted hyperboloids, settling the open issue raised in [11] of whether normal parameterizations for these varieties exist.

Variety	Equation	Parameterization	Missing Points
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$	$\left(\frac{a(s^2-1)}{s^2+1}, \frac{2bs}{a^2+1}\right)$	$(a,0)$
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$	$\left(\frac{2as}{s^2+t^2+1}, \frac{2bt}{s^2+t^2+1}, \frac{c(s^2+t^2-1)}{s^2+t^2+1}\right)$	$(0,0,c)$
Hyperboloid (1S)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$	$\left(\frac{a(s^2-t^2+1)}{s^2+t^2-1}, \frac{2bts}{s^2+t^2-1}, \frac{2cs}{s^2+t^2-1}\right)$	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0, z=0\right\} \setminus \{(-a,0,0)\}$
Hyperboloid (2S)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0$	$\left(\frac{2as}{s^2+t^2-1}, \frac{2bs}{s^2+t^2-1}, \frac{c(s^2+t^2+1)}{s^2+t^2-1}\right)$	$(0,0,c)$
Steiner	$x^2y^2+y^2z^2+x^2z^2-2xyz=0$	$\left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1} \frac{2st}{s^2+t^2+1}\right)$	$(0,0,r), r \in (-1,1), r \neq 0$

Table 1: Real parametric varieties and their missing points

Variety	Normal Parameterization
Ellipse	$\left(\frac{-a(u^4-3u^2+1)}{u^4-u^2+1}, \frac{2b(u-u^3)}{u^4-u^2+1}\right)$
Ellipsoid	$\left(\frac{2au(1-u^2-v^2)}{v^4+2u^2v^2-v^2+u^4-u^2+1}, \frac{2bv(1-u^2-v^2)}{v^4+2u^2v^2-v^2+u^4-u^2+1}, \frac{-c(v^4+2u^2v^2-3v^2+u^4-3u^2+1)}{v^4+2u^2v^2-v^2+u^4-u^2+1}\right)$
Hyperboloid (1S)	$\left(\frac{a(v^4+2u^2v^2-3v^2+u^4-u^2+1)}{-(v^4+2u^2v^2-3v^2+u^4-3u^2+1)}, \frac{2buv}{-(v^4+2u^2v^2-3v^2+u^4-3u^2+1)}, \frac{2cu(1-u^2-v^2)}{-(v^4+2u^2v^2-3v^2+u^4-3u^2+1)}\right)$
Hyperboloid (2S)	$\left(\frac{a(v^4+2u^2v^2-3v^2+u^4-u^2+1)}{-(v^4+2u^2v^2-3v^2+u^4-3u^2+1)}, \frac{2buv}{-(v^4+2u^2v^2-3v^2+u^4-3u^2+1)}, \frac{2cu(1-u^2-v^2)}{-(v^4+2u^2v^2-3v^2+u^4-3u^2+1)}\right)$
Steiner	$\left(\frac{-2uv^2-2u^3+2u}{v^4+2u^2v^2-v^2+u^4-u^2+1}, \frac{-2v^3-2u^2v+2v}{v^4+2u^2v^2-v^2+u^4-u^2+1}, \frac{2uv}{v^4+2u^2v^2-v^2+u^4-u^2+1}\right)$

Table 2: Normal parameterizations of some varieties

COROLLARY 3 *Using theorem 2, we can compute normal parameterizations for the ellipse, ellipsoid, hyperboloid of one sheet, and hyperboloid of two sheets.*

The equations of several varieties are given in Table 1. A rational parameterization for each variety is listed, as well as the missing points of each parameterization. The missing points can be computed either using the general machine computations of [11], but for these particular varieties direct, elementary arguments suffice [20].

In Table 2, normal parameterizations of these varieties are given; one for the ellipse is found in [11], all those for surfaces are new. For the ellipse, the parameters can be restricted to $|u| \leq 1$, and for the quadric surfaces they can be restricted to $u^2+v^2 \leq 1$. The parameterizations map these bounded domain regions onto the entire variety, without missing any points.

Examples. The upper half of Figure 3 shows a normal ellipsoid parameterization graphed over the parameter region $u^2+v^2 \leq 1$. Likewise, the lower half shows a normal Steiner surface parameterization. The parameters of both are restricted to the unit disk in $\mathcal{R}^2$. The left-hand side of each figure is unshaded to emphasize that no parts of the surface are missing (compare these to Figure 1).

5 Conclusions and future work

In this paper we have presented two ways by which infinite parameter values can be avoided when dealing with parametric curves and surfaces. The results were presented for parametric varieties of any dimension, and were used to solve the open problem of computing quadric surface parameterizations that do not have any missing points.

These results have been applied as a first step in the robust display of arbitrary real parametric curves and surface [20]. Another application is a first step towards exactly representing a arbitrary real parametric curve or surface in piecewise rational Bernstein-Bezier form, with positive weights.

Future directions for research include examining the other "missing points" problem (due to domain base points) in more detail, and also considering the issue of computing those real points on a real parametric variety that do not correspond to any real parameter value, but to a complex parameter value.

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