Modeling with Cubic A-Patches

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We present a sufficient criterion for the Bernstein Bezier (BB) form of a trivariate polynomial within a tetrahedron, such that the real zero contour of the polynomial defines a smooth and single-sheeted algebraic surface patch. We call this an A-patch. We present algorithms to build a mesh of cubic A-patches to interpolate a given set of scattered point data in three dimensions, respecting the topology of any surface triangulation T of the given point set. In these algorithms we first specify "normals" on the data points, then build a simplicial hull consisting of tetrahedra surrounding the surface triangulation T, and finally construct cubic A-patches within each tetrahedron. The resulting surface constructed is C^1 (tangent plane) continuous and single sheeted in each of the tetrahedra. We also show how to adjust the free parameters of the A-patches to achieve both local and global shape control.

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1. INTRODUCTION

The importance of implicit surface representation in modeling geometric objects or reconstructing the image to scattered data has been described in various papers (see, e.g., Bajaj 1993, Dahmen and Thamm-Schaar 1993, Guo 1991a, Lodha 1992, and Sederberg 1985). The main advantages of implicit surface over its parametric counterpart are that (1) the set of algebraic surfaces are closed under basic modeling operations such as offset and intersection, often required in a solid modeling system. For example, the offset of a parametric surface may not be parametric, but is always algebraic and has an implicit representation. (2) For the same polynomial of degree n,

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implicit algebraic surfaces have more degrees of freedom $\left(= \binom{n+3}{3} - 1\right)$ compared with rational parametric surface $\left(\le 4\binom{n+2}{2} - 1\right)$ of the same degree. Hence, implicit algebraic surfaces are more flexible for approximating a complicated surface with a fewer number of pieces or for achieving a higher order of smoothness. However, the main shortcoming held against the popular use of implicit surfaces is that the representation being multivalued may cause the real zero-contour surface to have multiple sheets, self-intersections, and several other undesirable singularities.

In Section 3 we present a sufficient criterion for the Bernstein–Bezier (BB) form of a trivariate polynomial within a tetrahedron such that the real zero contour of the polynomial is a smooth (nonsingular), single-sheeted algebraic surface. We call this an *A-patch*. In Section 4 we describe how to build a simplicial hull consisting of tetrahedra surrounding a surface triangulation T of the set of scattered data points in 3-D. We then show in Section 5 how a mesh of cubic A-patches can be used to construct a C^1 interpolatory surface, respecting the topology of the surface triangulation T. In Section 6 we show how to adjust the free parameters of the A-patches to achieve both local and global shape control. This C^1 cubic A-patch fitting algorithm is quite appropriate for free-form design. In analogy to the final smoothing of an artist's rough sketches, complicated smooth models can be directly formed by first creating a rough polyhedral model of the desired object and then using the fitting algorithms to produce a C^1 smooth solid with extra local and global parameters for fine shape control. Proofs of all the theorems and lemmas are given in the Appendix.

1.1 Related Prior Work

The work of characterizing the BB form of polynomials within a tetrahedron, such that the zero contour of the polynomial is a single-sheeted surface within the tetrahedron, has been attempted in the past. Sederberg [1985] showed that, if the coefficients of the BB form of the trivariate polynomial on the lines that parallel one edge, say, L, of the tetrahedron all increase (or decrease) monotonically in the same direction, then any line parallel to L will intersect the zero-contour algebraic surface patch at most once. Guo [1991a] treated the same problem by enforcing monotonicity conditions on a cubic polynomial along the direction from one vertex to a point of the opposite face of the vertex. From this he derived a condition $a_{\lambda - e_1 + e_4} - a_\lambda \ge 0$ for all $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ with $\lambda_1 \ge 1$, where a_λ are the coefficients of the cubic in BB form and e_i is the ith-unit vector. This condition is difficult to satisfy in general, and even if this condition is satisfied, one still cannot avoid singularities on the zero contour. Our condition of a smooth, single-sheeted zero contour in Theorem 3.2 generalizes Sederberg's condition and provides an efficient way of generating A-patches.

The second problem we consider is how to join a collection of A-patches to form a C^1 smooth surface interpolating scattered data points and respecting the topology of a given surface triangulation T of the points. For this problem, prior approaches have been given by Dahmen [1989] using quadric patches, Dahmen and Thamm-Schaar [1993] and Guo [1991a, 1991b] using cubic patches, and Bajaj and Ihm [1992] using quintic patches for convex triangulations and degree-seven patches for arbitrary surface triangulations T. All of these papers provide heuristics to overcome the multiple-sheeted and singularity problems of implicit patches. In this paper our cubic A-patches are guaranteed to be nonsingular and single sheeted within each tetrahedron.

While the details of the methods of Dahmen and Thamm-Schaar [1993] and Guo [1991b] differ somewhat, they both use the scheme of Dahmen [1989] of building a surrounding simplicial hull (consisting of a series of tetrahedra) of the given triangulation T. Such a simplicial hull is nontrivial to construct for triangulations, and none of the papers [Dahmen 1989; Dahmen and Thamm-Schaar 1993; Guo 1991a, 1991b] enumerate the different exceptional cases (possible even for convex triangulations) nor provide solutions to overcoming them. We too use the simplicial hull approach in this paper, but enumerate the exceptional situations and provide some heuristic strategies for rectifying them.

Guo [1991b] used a Clough–Tocher split [Clough and Tocher 1965] and subdivided each face tetrahedron of the simplicial hull, hence utilizing three patches per face of T. In this paper we consider the computed "normals" at the given data points, and distinguish between "convex" and "nonconvex" faces and edges of the triangulation. These concepts are formally defined in Section 4. We use a single cubic A-patch per face of T except for the following two special cases: (1) For a nonconvex face, if, additionally, the three inner products of the face normal and its three adjacent face normals have different signs, then one needs to subdivide the face using a single Clough–Tocher split, yielding C^1 continuity with the help of three cubic A-patches for that face. (2) Furthermore, for coplanar adjacent faces of T, we show that the C^1 conditions cannot be met using a single cubic A-patch for each face. Hence, we again use Clough–Tocher splits for the pair of coplanar faces, yielding C^1 continuity with the help of three cubic A-patches per face. See also the examples and figures in Section 7, where the savings in patches becomes evident.

Related papers that approximate scattered data using implicit algebraic patches are [Bajaj 1992, Lodha 1992, and Moore and Warren 1991]. A classification of data fitting using parametric surface patches is given in Peters [1990].

2. NOTATION AND PRELIMINARY DETAILS

Problem. Given a list of data points $P = \{p_1, \dots, p_k\} \in \mathbb{R}^3$ and a surface triangulation T of these points, construct a mesh of low-degree algebraic surfaces such that the composite surface is single-sheeted C^1 continuous and has the same topology as T.

Convex hull and affine hull. Let $\{p_1, \dots, p_j\} \in \mathbb{R}^3$ with $j \leq 4$. Then the *convex hull* of these points is defined by $[p_1 p_2 \cdots p_j] = \{p \in \mathbb{R}^3 : p$

$= \sum_{i=1}^{j} \alpha_i p_i,\ \alpha_i \geq 0,\ \sum_{i=1}^{j} \alpha_i = 1\}$, and the *affine hull* is defined by $\langle p_1 p_2 \cdots p_j \rangle = \{p \in \mathbb{R}^3 : p = \sum_{i=1}^{j} \alpha_i p_i,\ \sum_{i=1}^{j} \alpha_i = 1\}$. The interior of the convex hull $[p_1 p_2 \cdots p_j]$ is denoted by $(p_1 p_2 \cdots p_j) = \{p \in \mathbb{R}^3 : p = \sum_{i=1}^{j} \alpha_i p_i,\ \alpha_i > 0,\ \sum_{i=1}^{j} \alpha_i = 1\}$.

Bernstein–Bezier (BB) form. Let $p_1, p_2, p_3, p_4 \in \mathbb{R}^3$ be affine independent. Then the tetrahedron with vertices p_1, p_2, p_3, and p_4 is $V = [p_1 p_2 p_3 p_4]$. For any $p = \sum_{i=1}^{4} \alpha_i p_i \in V$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is the barycentric coordinate of p. Let $p = (x, y, z)^T$ and $p_i = (x_i, y_i, z_i)^T$. Then the barycentric coordinates relate to the Cartesian coordinates via the following relation:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}. \tag{2.1}$$

Any polynomial $f(p)$ of degree n can be expressed as BB form over V as $f(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$, $\lambda \in \mathcal{Z}_+^4$, where

$$B_\lambda^n(\alpha) = \frac{n!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!} \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \alpha_3^{\lambda_3} \alpha_4^{\lambda_4}$$

is Bernstein polynomial, $|\lambda| = \sum_{i=1}^{4} \lambda_i$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T = \sum_{i=1}^{4} \alpha_i e_i$ is a barycentric coordinate of p, $b_\lambda = b_{\lambda_1\lambda_2\lambda_3\lambda_4}$ (as a subscript, we simply write λ as $\lambda_1\lambda_2\lambda_3\lambda_4$) are called *control points*, and $\mathcal{Z}_+^4$ stands for the set of all four-dimensional vectors with nonnegative integer components. The following basic facts about the BB form will be used in this paper. The first is derived from the directional derivative formulas (see Farin 1990):

LEMMA 2.1. *If* $f(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$, *then*

$$b_{(n-1)e_i+e_j} = b_{ne_i} + \frac{1}{n}(p_j - p_i)^T \nabla f(p_i), \qquad j = 1,2,3,4;\ \ j \neq i, \tag{2.2}$$

where

$$\nabla f(p) = \left[\frac{\partial f(p)}{\partial x} \frac{\partial f(p)}{\partial y} \frac{\partial f(p)}{\partial z}\right]^T.$$

Formula (2.2) will be used to determine the control points around a vertex from the given normal at that vertex.

LEMMA 2.2 [Farin 1990]. *Let* $f(p) = \sum_{|\lambda|=n} a_\lambda B_\lambda^n(\alpha)$ *and* $g(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ *be two polynomials defined on two tetrahedra* $[p_1 p_2 p_3 p_4]$ *and* $[p_1' p_2 p_3 p_4]$, *respectively. Then*

(i) f *and* g *are* C^0 *continuous at the common face* $[p_2 p_3 p_4]$ *if and only if (iff)*

$$a_\lambda = b_\lambda, \qquad \text{for any} \quad \lambda = 0\lambda_2\lambda_3\lambda_4, \quad |\lambda| = n. \tag{2.3}$$

(ii) *f and g are C^1 continuous at the common face* $[p_2 p_3 p_4]$ *iff* (2.3) *holds and*

$$b_{1\lambda_2\lambda_3\lambda_4} = \beta_1 a_{1\lambda_2\lambda_3\lambda_4} + \beta_2 a_{0\lambda_2\lambda_3\lambda_4+0100} + \beta_3 a_{0\lambda_2\lambda_3\lambda_4+0010} + \beta_4 a_{0\lambda_2\lambda_3\lambda_4+0001}, \tag{2.4}$$

where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^T$ *are defined by the relation* $p_1' = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + \beta_4 p_4$, $|\beta| = 1$.

Relation (2.4) is called *coplanar condition*.

Degree elevation. The polynomial $f(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ can be written as one of dgree $n+1$ (see, e.g., Farin 1990): $f(p) = \sum_{|\lambda|=n+1} (Eb)_\lambda B_\lambda^{n+1}(\alpha)$, $\lambda \in \mathbb{Z}_+^4$, where $(Eb)_\lambda = 1/(n+1) \sum_{i=1}^4 \lambda_i b_{\lambda - e_i}$.

Variation diminishing property [Farin 1990, p. 54]. Let $y(t) = \sum_{i=0}^n b_i B_i^n(t)$; then $y(t)$ has no more intersections (counting the multiplicities) with any line than does the polygon $\{i/n, b_i\}_{i=0}^n$ in [0, 1].

Transformation. Since $\sum_{i=1}^4 \alpha_k = 1$, we have from (2.1) that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 - x_4 & x_2 - x_4 & x_3 - x_4 \\ y_1 - y_4 & y_2 - y_4 & y_2 - y_4 \\ z_1 - z_4 & z_2 - z_4 & z_2 - z_4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} x_4 \\ y_4 \\ z_4 \end{bmatrix} = A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} x_4 \\ y_4 \\ z_4 \end{bmatrix}. \tag{2.5}$$

Let $f(x, y, z) = g(\alpha_1, \alpha_2, \alpha_3)$. Then it is easy to check that

$$\nabla f(x, y, z) = (A^{-1})^T \nabla g(\alpha_1, \alpha_2, \alpha_3). \tag{2.6}$$

Therefore, the surface $f(x, y, z) = 0$ is smooth (i.e., $\nabla f(x, y, z) \neq 0$) iff the surface $g(\alpha_1, \alpha_2, \alpha_3) = 0$ is smooth (i.e., $\nabla g(\alpha_1, \alpha_2, \alpha_3) \neq 0$). This means that the smoothness problem of the surface $f(x, y, z) = 0$ can be treated directly in its barycentric form.

3. SUFFICIENT CONDITIONS OF AN A-PATCH

Let $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ be a given polynomial of degree n on the simplex (tetrahedron) $S = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T \in \mathbb{R}^4 : \sum_{i=1}^4 \alpha_i = 1, \alpha_i \geq 0\}$. The surface patch within the simplex is defined by $S_F \subset S : F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$. The following two conditions on the trivariate BB form will be used in this paper:

Smooth vertices condition. For each $i (1 \leq i \leq 4)$, there is at least one nonzero $b_{\lambda_1\lambda_2\lambda_3\lambda_4}$ for $\lambda_i \geq n - 1$.

Smooth edges condition. For each pair $(i, j)(1 \leq i, j \leq 4, i \neq j)$, there either is at least one nonzero $b_{me_i + (n-m)e_j}$, for $m = 0, 1, \ldots, n$, or the polynomials $\sum_{m=0}^{n-1} b_{me_i+(n-1-m)e_j+e_k} B_m^{n-1}(t)$ and $\sum_{m=0}^{n-1} b_{me_i+(n-1-m)e_j+e_l} B_m^{n-1}(t)$ have no common zero in [0, 1], for distinct i, j, k, l.

If the surface S_F contains a vertex/edge, then by the formulas of directional derivatives (see Farin 1990, p. 312), it is easy to show that the surface is smooth there if the smooth vertex/edge conditions above are satisfied.

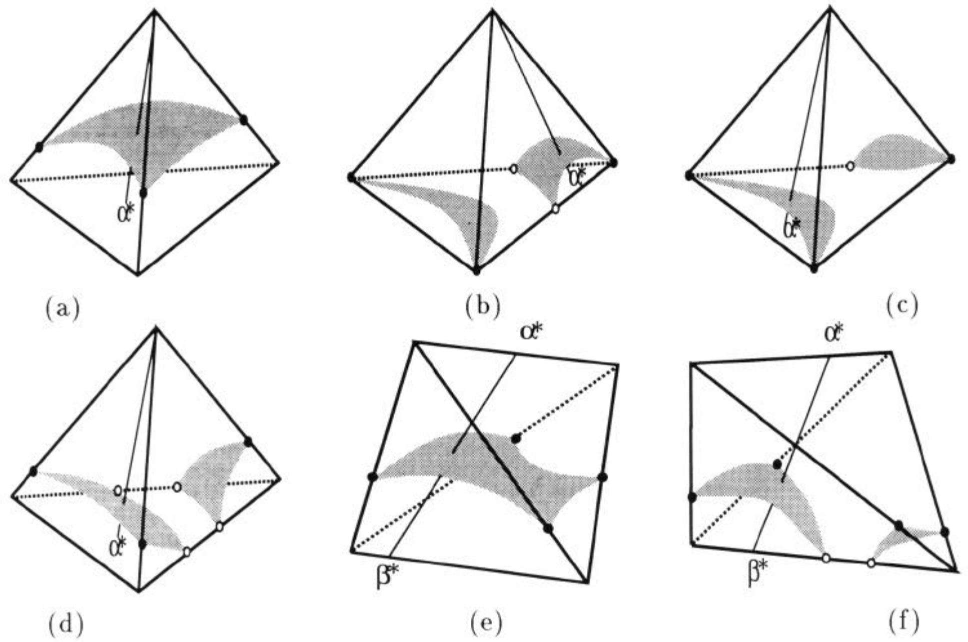


Fig. 1. Three-sided (a, b, c and d) and four-sided patches (e and f). Some of them are disconnected. The filled vertices mark the boundaries of the patches.

Definition 3.1 *Three-sided patch.* Let the surface patch S_F be smooth on the boundary of the tetrahedron S. If any open line segment $(e_j,\ \alpha^*)$ with $\alpha^* \in S_j = \{(\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)^T : \alpha_j = 0,\ \alpha_i > 0,\ \sum_{i \neq j} \alpha_i = 1\}$ intersects S_F at most once (counting multiplicities), then we call S_F a *three-sided j-patch* (see Figure 1).

Definition 3.2 *Four-sided patch.* Let the surface patch S_F be smooth on the boundary of the tetrahedron S. Let $(i,\ j,\ k,\ l)$ be a permutation of $(1, 2, 3, 4)$. If any open line segment $(\alpha^*,\ \beta^*)$ with $\alpha^* \in (e_i e_j)$ and $\beta^* \in (e_k e_l)$ intersects S_F at most once (counting multiplicities), then we call S_F a *four-sided ij-kl-patch* (see Figure 1).

It is easy to see that, if S_F is a four-sided *ij-kl*-patch, it is then also a *ji-lk*-patch, a *lk-ji*-patch, and so on. The Appendix contains proofs of the following lemmas and theorems:

LEMMA 3.1. *The three-sided j-patch and the four-sided ij-kl-patch are smooth (nonsingular).*

THEOREM 3.2. *Let* $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ *satisfy the smooth vertex and smooth edge conditions and* $j(1 \leq j \leq 4)$ *be a given integer. If there exists an integer* $k(0 \leq k < n)$ *such that*

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \geq 0, \qquad \lambda_j = 0, 1, \ldots, k-1, \tag{3.1}$$

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \leq 0, \qquad \lambda_j = k+1, \ldots, n, \tag{3.2}$$

and $\sum_{\substack{|\lambda| = n \\ \lambda_j = 0}} b_\lambda > 0$ *if* $k > 0$, $\sum_{\substack{|\lambda| = n \\ \lambda_j = m}} b_\lambda < 0$ *for at least one* $m (k < m \le n)$, *then* S_F *is a three-sided j-patch.*

THEOREM 3.3. *Let* $F(\alpha) = \sum_{|\lambda| = n} b_\lambda B_\lambda^n(\alpha)$ *satisfy the smooth vertex and smooth edge conditions, and let* (i, j, k, l) *be a permutation of* $(1, 2, 3, 4)$. *If there exists an integer* $k (0 \le k < n)$ *such that*

$$b_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \ge 0, \qquad \lambda_i + \lambda_j = 0, 1, \ldots, k - 1, \tag{3.3}$$

$$b_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \le 0, \qquad \lambda_i + \lambda_j = k + 1, \ldots, n, \tag{3.4}$$

and $\sum_{\substack{|\lambda| = n \\ \lambda_i + \lambda_j = 0}} b_\lambda > 0$ *if* $k > 0$, $\sum_{\substack{|\lambda| = n \\ \lambda_i + \lambda_j = m}} b_\lambda < 0$ *for at least one* $m (k < m \le n)$, *then* S_F *is a four-sided ij-kl-patch.*

Note. The conditions on the coefficients b_λ in Theorems 3.2 and 3.3 are sufficient, but not necessary. For example, if we want some $B_l < 0$, it is not necessary to let every $b_\lambda < 0$, for $|\lambda| = n$, $\lambda_4 = l$.

Some properties of A-patches

(a) For a three-sided j-patch, if $b_\lambda = 0$ for $\lambda = (n - l)e_m + le_j$, $l = 0, 1, \ldots, k (m \ne j, k < n)$, and $b_\lambda \ne 0$ for $\lambda = (n - 1)e_m + e_s$, $s \ne j, m$, then the edge $[e_j e_m]$ is tangent with S_F at e_m with multiplicities k. See also Figure 2a.

(b) For a four-sided ij-kl-patch, if $b_\lambda = 0$ for $\lambda = (n - q_1 - q_2)e_k + q_1 e_i + q_2 e_j$, $q_1 + q_2 = 0, 1, \ldots, s$, and $b_\lambda \ne 0$ for $\lambda = (n - 1)e_k + e_l$, then S_F is tangent s times with face $[e_i e_j e_k]$ at e_k.

Note that a four-sided patch may degenerate into a two-sided patch; see Figure 2b. However, we do not need to treat the degenerate patches any different, but consider it to be a special four-sided patch.

(c) For a three-sided j-patch, if $b_\lambda = 0$ for $\lambda = (n - m)e_i + me_k$, $m = 0, 1, \ldots, n$, then S_F contains the edge $[e_i, e_k]$. Furthermore, if $b_\lambda = 0$ for $\lambda = (n - m - 1)e_i + me_k + e_j$, $m = 0, 1, \ldots, n - 1$, then the S_F is tangent with the face $[e_i e_j e_k]$. See also Figure 3a and b.

(d) For a three-sided j-patch, any point $P \in S_F$ can be mapped to a triple $(\alpha_i, \alpha_k, \alpha_l)$, $\alpha_i + \alpha_k + \alpha_l = 1$, $\alpha_i, \alpha_k, \alpha_l \ge 0$ or a point $\alpha^* \in S_j = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T : \alpha_j = 0\}$, Furthermore, there exists a one to one mapping between S_F and $S_j' = \{\alpha^* : \alpha^* \in S_j, F(e_j) \cdot F(a*) \le 0\}$.

(e) For a four-sided ij-kl-patch, any point $P \in S_F$ can be mapped to a tuple (α_i, α_k), $0 \le \alpha_i \le 1$, $0 \le \alpha_k \le 1$, or two points $\alpha^* \in (e_i e_j) = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T : \alpha_k = \alpha_l = 0\}$ and $\beta^* \in (e_k e_l) = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T : \alpha_i = \alpha_j = 0\}$. Furthermore, there exists a one-to-one mapping between S_F and $\{(\alpha_i, \beta_k)^T : F(\alpha^*) \cdot F(\beta^*) \le 0\}$. If $F(\alpha^*) = 0$, S_F is degenerated and all the points with the same α_k collapse into one point.

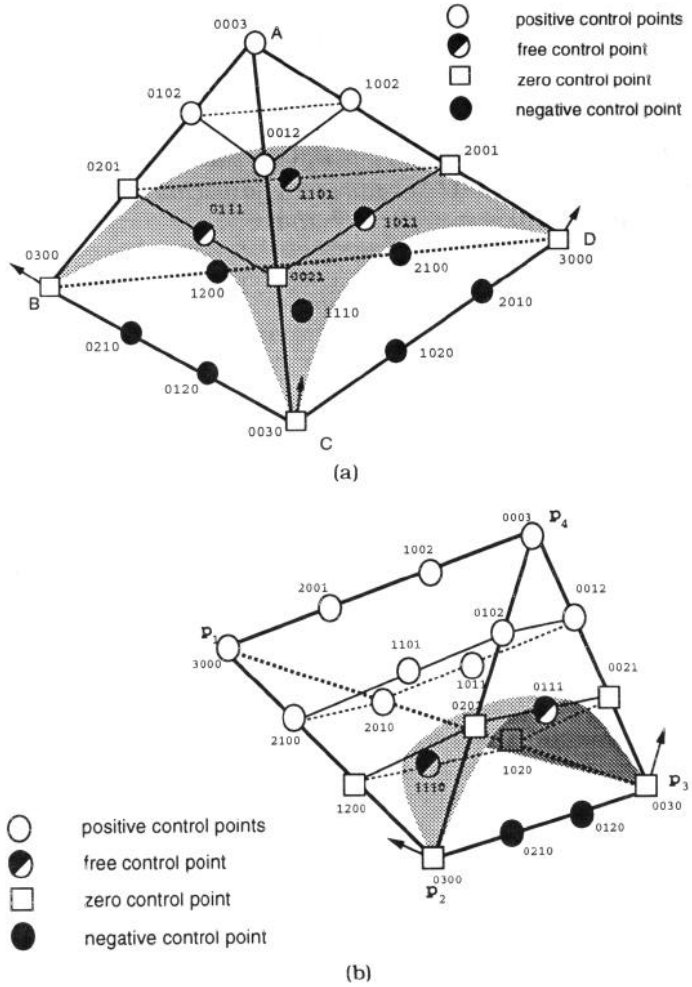


Fig. 2. (a) Three-sided patch tangent at p_1, p_2, p_3. (b) Degenerate four-sided patch tangent to face $[p_1 p_2 p_4]$ at p_2 and to face $[p_1 p_3 p_4]$ at p_3.

Hence a three-sided patch can be mapped into a triangular domain while a four-sided patch can be mapped into a quadrilateral domain. This observation gives rise to the terms *three-sided patch* and *four-sided patch*.

Note that smooth three-sided or smooth four-sided patches are not necessarily connected within a single tetrahedron. Figure 1 shows some examples. Subsequent sections detail how a combination of smooth A-patches are pieced together to form a C^1 smooth global surface.

4. NORMALS AND THE SIMPLICIAL HULL

For the given point set $P = \{p_1, \dots, p_k\} \in \mathbb{R}^3$ and their surface triangulation T, we first construct a normal set $N = \{n_1, \dots, n_k\} \in \mathbb{R}^3$ for P. That is, for

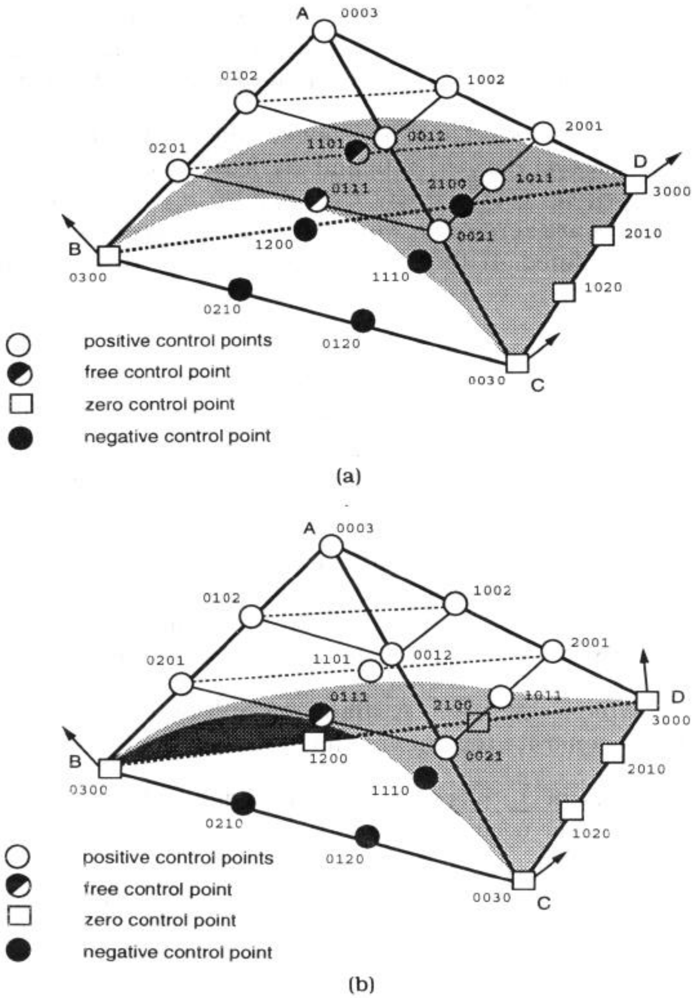


Fig. 3. (a) Three-sided patch interpolating edge $[p_2 p_3]$. (b) Three-sided patch interpolating edges $[p_2 p_3]$ and $[p_1 p_3]$.

each point p_i, we associate a normal n_i. We will force the constructed surface to interpolate points p_i and at each point have a normal n_i for $i = 1, \ldots, k$. These normals therefore also provide a mechanism to control the shape of the C^1 interpolating surface. Common approaches to construct these normals at a point p_i include (1) an average of the face normals of the incident faces, and (2) the gradient of a local spherical fit to the surface triangulation at each vertex. Computing an optimal normal assignment is yet an unsolved problem, and we are experimenting with different local and global normal selections schemes [Bajaj 1992; Pottmann 1992; Moreton 1993]. Of course, at times the data set can have prespecified normals, and this too can be the input of the C^1 fitting algorithm.

Without loss of generality, we assume that the assigned normals all point to the same side of T. If T is a closed surface triangulation (a simplicial polyhedron), then we assume that the normals all point to the exterior.

Definition 4.1 *Convex edge and nonconvex edge.* Let $[p_i p_j]$ be an edge of T. If $(p_j - p_i)^T n_i (p_i - p_j)^T n_j \geq 0$ and at least one of $(p_j - p_i)^T n_i$ and $(p_i - p_j)^T n_j$ is positive, then we say the edge $[p_i p_j]$ is *negative convex*. If both of the numbers are zero, then we say it is *zero convex*. A *positive convex* edge is similarly defined. If $(p_j - p_i)^T n_i (p_i - p_j)^T n_j < 0$, then we say the edge is *nonconvex*.

Definition 4.2 *Convex face and nonconvex face.* Let $[p_i p_j p_k]$ be a face of T. If its three edges are nonnegative (positive or zero) convex and at least one of them is positive convex, then we say the face $[p_i p_j p_k]$ is *positive convex*. If all of the three edges are zero convex, then we label the face as *zero convex*. A *negative convex* face is similarly defined. All of the other cases $[p_i p_j p_k]$ are labeled as *nonconvex*.

Note that here we are overloading the term *convex* to characterize the relations between the normals and edges of faces. We distinguish between convex and nonconvex faces in the simplicial hull below, where we build one tetrahedron for convex faces and double tetrahedra for nonconvex faces.

Definition 4.3 *Simplicial hull.* A simplicial hull of T, denoted by $\sum$, is a collection of nondegenerate tetrahedra that satisfies the following:

(1) Each tetrahedron in $\sum$ has either a single edge of T (then it is called an *edge tetrahedron*) or a single face of T (then it is called a *face tetrahedron*).
(2) For each face of T, there is (are) only one or two face tetrahedra in $\sum$ if the face is convex or nonconvex.
(3) Two face tetrahedra that share a common edge do not intersect anywhere else. This condition is referred to as *nonintersection*.
(4) For each edge, there is (are) only one or two pair(s) of common face sharing edge tetrahedra in $\sum$ if the edge is convex or nonconvex such that the pair(s) fill the region between the two adjacent face tetrahedra in the same side of T.
(5) For each vertex, the tangent plane defined by the vertex normal is contained in all of the tetrahedra containing the vertex. This condition is called *tangent plane containment*.

Note that, for a given surface triangulation with normal assignments, T. there may exist infinitely many simplicial hulls, or no simplicial hull may exist. We now describe a scheme for constructing a simplicial hull for the surface triangulation T and prescribed vertex normal assignment. We also enumerate the exceptional configurations where a simplicial hull of T is difficult and then provide a solution for constructing the simplicial hull for a locally modified T.

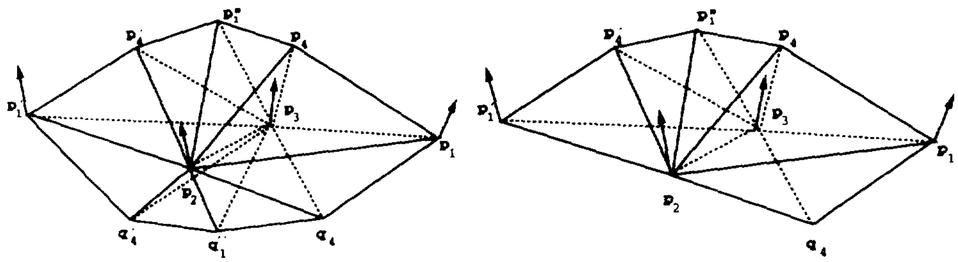


Fig. 4. The construction of double tetrahedra for adjacent nonconvex/nonconvex faces and convex/nonconvex faces.

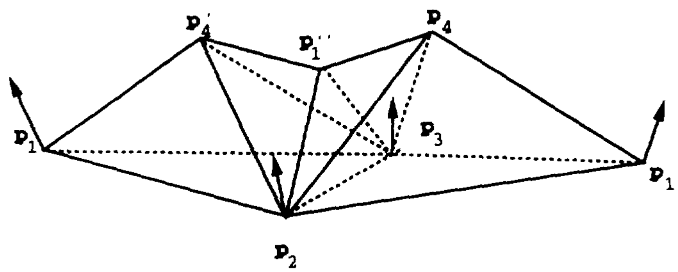


Fig. 5. Construction of single tetrahedra for adjacent convex/convex faces.

(1) *Build face tetrahedra.* For each face $F = [p_1 p_2 p_3]$ of T, let L be a straight line that is perpendicular to the face F and that passes through the center of the inscribed circle of F. Then choose points p_4 and/or q_4 off each side of F to be the farthermost intersection points between L and the tangent planes of the vertices of the face. If F is a nonconvex face, two face tetrahedra $[p_1 p_2 p_3 p_4]$ and $[p_1 p_2 p_3 q_4]$ are formed (double tetrahedra). If F is positive convex, then p_4 is chosen on the same side as the direction of the normals, and a single face tetrahedron $[p_1 p_2 p_3 p_4]$ is formed. If F is negative convex, then q_4 is chosen on the the opposite side of the normals, and again a single face tetrahedron $[p_1 p_2 p_3 q_4]$ is formed. Figure 4 shows the cases where at least one of the two adjacent faces is nonconvex, and Figure 5 shows the case where both faces are convex.

A sufficient condition for constructing face tetrahedra with tangent plane containment is that the angle of the assigned normal n_i at each vertex p_i with each of the surrounding face's normals is less than $\pi/2$. If this condition is not met, then an exception occurs, and we term the vertex as *sharp*. See Figure 6a.

A sufficient condition for adjacent face tetrahedra to be nonintersecting is as follows: For two adjacent faces $F = [p_1 p_2 p_3]$ and $F' = [p'_1 p_2 p_3]$, the angle between them, denoted as $\angle FF'$, is defined as the outer dihedral angle if the edge between F and F' is negative convex and inner dihedral angle

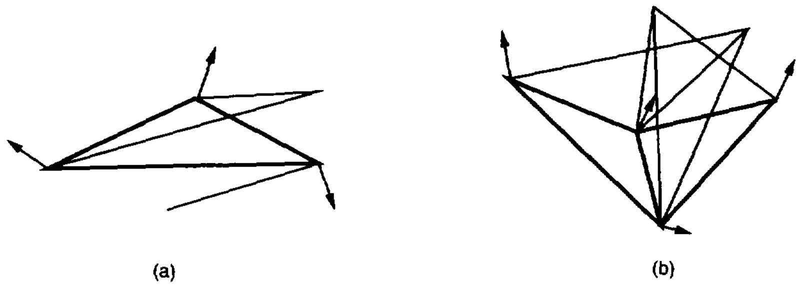


Fig. 6. (a) No tangent plane containment. (b) Self-intersecting tetrahedra.

otherwise. For $[p_2 p_3]$, the common edge between F and F', let $[p_1 p_2 p_3 p_4]$ and $[p'_1 p_2 p_3 p'_4]$ be the face tetrahedra, respectively. Then the two tetrahedra are nonintersecting if the angles $\angle[p_4 p_2 p_3][p_1 p_2 p_3] < \frac{1}{2}\angle FF'$ and $\angle[p'_4 p_2 p_3][p'_1 p_2 p_3] < \frac{1}{2}\angle FF'$. If this condition is not met, then an exception may occur, and we term the common edge $[p_2 p_3]$ as *sharp*. See Figure 6b.

A heuristic strategy that rectifies the *sharp* edge and *sharp* vertex configurations is a local retriangulation of the original surface triangulation T. This strategy has worked well in several of the smoothing examples we have performed. An exact algorithm based on subdivision is given in Bajaj et al. [1994].

(i) *Sharp edge problem.* Let $[p_1 p_2]$ be a sharp edge (see Figure 7a), and let $[p_i p_{ij}]$ $(i = 1, 2;\ j = 1, 2, \dots, k_i)$ be the remaining surrounding edges of p_i in adjacency order. Take two spheres $S(p_i, r_i)$ with centers p_i and radius r_i, where r_i are positive numbers that are less than half of the surrounding edge's lengths $\|p_i - p_{ij}\|$. The smaller r_i is, the sharper the constructed surfaces around edge $[p_1 p_2]$ are. Let q_{ij} be the intersection points of $S(p_i, r_i)$ and $[p_i p_{ij}]$. Then $q_{i1}, q_{i2}, \dots, q_{ik_i}$ form two closed polygons, $p_{ij}, p_{ij+1}, q_{ij+1}, q_{ij}$ form a four-sided closed polygon, and, finally, $q_{11}, q_{21}, q_{2k_2}, q_{1k_1}$ form another four-sided closed polygon. Triangulate these polygons (the dotted line in Figure 7a) by connecting adjacent edges of the polygons in the least inner angle order.

(ii) *Sharp vertex problem.* Let p_1 be a sharp vertex (see Figure 7b), and let $[p_1 p_{1j}]$ $(j = 1, 2, \dots, k)$ be the surrounding edges of p_1 in adjacency order. Take a sphere $S(p_1, r)$ with center p_1 and radius r, where r is a positive number that is less than half of the surrounding edge's lengths $\|p_1 - p_{1j}\|$. The smaller r is, the sharper the constructed surfaces around vertex p_1 are. Let q_{1j} be the intersection points of $S(p_1, r)$ and $[p_1 p_{1j}]$. Then $q_{11}, q_{12}, \dots, q_{1k}$ form a closed polygon, and $p_{1j}, p_{1j+1}, q_{1j+1}, q_{1j}$ form a four-sided closed polygon. Triangulate these polygons (the dotted line in Figure 7b) by connecting the adjacent edges of the polygon in the least inner angle.

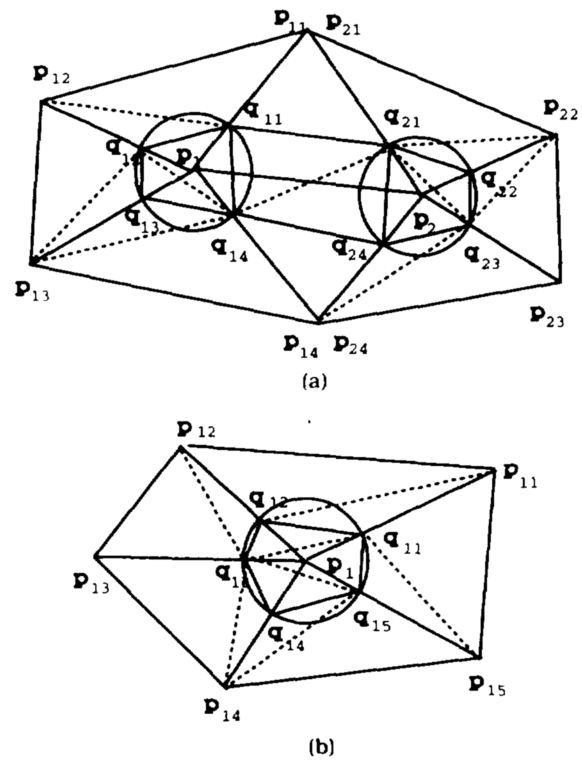


Fig. 7. Retriangulation of (a) sharp edge and (b) sharp vertex.

(2) *Build edge tetrahedra.* Let $[p_2 p_3]$ be an edge of T, and let $[p_1 p_2 p_3]$ and $[p'_1 p_2 p_3]$ be the two adjacent faces. Let $[p_1 p_2 p_3 p_4]$ and/or $[p_1 p_2 p_3 q_4]$, and $[p'_1 p_2 p_3 p'_4]$ and/or $[p'_1 p_2 p_3 q'_4]$ be the face tetrahedra built for the faces $[p_1 p_2 p_3]$ and $[p'_1 p_2 p_3]$, respectively. Then, if edge $[p_2 p_3]$ is nonconvex, two pairs of tetrahedra need to be constructed. The first pair $[p''_1 p_2 p_3 p_4]$ and $[p''_1 p_2 p_3 p'_4]$ is between $[p'_1 p_2 p_3 p'_4]$ and $[p_1 p_2 p_3 p_4]$. The second pair $[q''_1 p_2 p_3 q_4]$ and $[q''_1 p_2 p_3 q'_4]$ is between $[p'_1 p_2 p_3 q'_4]$ and $[p_1 p_2 p_3 q_4]$. Here, $p''_1 \in (p_4 p'_4)$ or is above (p_4, p'_4), say,

$$p''_1 = \frac{(1-t)}{2}(p_2 + p_3) + \frac{t}{2}(p'_4 + p_4), \qquad t \geq 1,$$

so that p''_1 is above plane $[p_1 p_2 p_3]$ and plane $[p'_1 p_2 p_3]$. Similarly, $q''_1 \in (q_4 q'_4)$ or is below (q_4, q'_4), say,

$$q''_1 = \frac{(1-t)}{2}(p_2 + p_3) + \frac{t}{2}(q'_4 + q_4), \qquad t \geq 1,$$

so that q''_1 is below plane $[p_1 p_2 p_3]$ and plane $[p'_1 p_2 p_3]$. If edge $[p_2 p_3]$ is positive/negative convex, only the first/second pair above is needed. If the edge $[p_2 p_3]$ is zero convex, no tetrahedron is needed here. It should be noted that p_4 and p'_4 (q_4 and q'_4) are always visible.

5. CONSTRUCTION OF A C^1 INTERPOLATORY SURFACE USING CUBIC A-PATCHES

Having established a simplicial hull $\sum$ for the given surface triangulation T and a set of vertex normals N, we now construct a C^1 function f on the hull $\sum$ such that

$$f(p_i) = 0, \qquad \nabla f(p_i) = n_i, \qquad i = 1, 2, \dots, k, \tag{5.1}$$

and the zero contour of f within $\sum$ forms a C^1 continuous single-sheeted surface with the same topology as T.

5.1 The Construction of a Piecewise C^1 Cubic Function

The construction of the function f over two adjacent faces of T is divided into the following three cases:

(a) Both faces are nonconvex.
(b) Both faces are convex.
(c) One of them is convex, and the other is nonconvex.

(a) *Both faces are nonconvex.* Let $F = [p_1 p_2 p_3]$ and $F' = [p'_1 p_2 p_3]$ be two adjacent nonconvex faces. Then we have double tetrahedra $[p_1 p_2 p_3 p_4]$ and $[p_1 p_2 p_3 q_4]$ for F, and double tetrahedra $[p'_1 p_2 p_3 p'_4]$ and $[p'_1 p_2 p_3 q'_4]$ for F' (see Figure 8). Let

$$V_1 = [p_1 p_2 p_3 p_4], \qquad V_2 = [p'_1 p_2 p_3 p'_4], \qquad W_1 = [p''_1 p_2 p_3 p_4],$$
$$W_2 = [p''_1 p_2 p_3 p'_4], \qquad V'_1 = [p_1 p_2 p_3 q_4], \qquad V'_2 = [p'_1 p_2 p_3 q'_4],$$
$$W'_1 = [q''_1 p_2 p_3 q_4], \qquad W'_2 = [q''_1 p_2 p_3 q'_4],$$

and the cubic polynomials f_i over V_i, g_i over W_i, f'_i over V'_i, and g'_i over W'_i be expressed in BB forms with coefficients a^i_λ, b^i_λ, c^i_λ, and d^i_λ, $i = 1, 2$, respectively. Now we shall determine these coefficients.

C^0 Continuity. If two tetrahedra share a common face, we equate the control points of the associated cubic polynomials on the common face (see Lemma 2.2):

$$a^i_{\lambda_1\lambda_2\lambda_3 0} = c^i_{\lambda_1\lambda_2\lambda_3 0}, \qquad a^i_{0\lambda_2\lambda_3\lambda_4} = b^i_{0\lambda_2\lambda_3\lambda_4}, \qquad b^1_{\lambda_1\lambda_2\lambda_3 0} = b^2_{\lambda_1\lambda_2\lambda_3 0},$$
$$c^i_{0\lambda_2\lambda_3\lambda_4} = d^i_{0\lambda_2\lambda_3\lambda_4}, \qquad d^1_{\lambda_1\lambda_2\lambda_3 0} = d^2_{\lambda_1\lambda_2\lambda_3 0}.$$

Interpolation. Since zero contours of f_i and f'_i and g_i and g'_i pass through p_2 and p_3, $a^i_\lambda = b^i_\lambda = c^i_\lambda = d^i_\lambda = 0$ for $i = 1, 2$, and $\lambda = 0300, 0030$.

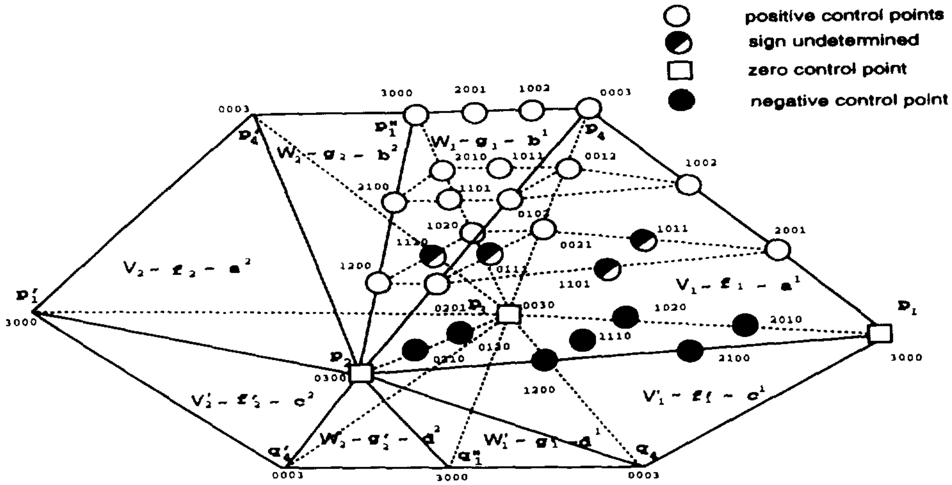


Fig. 8. Adjacent double tetrahedra, functions, and control points for two nonconvex adjacent faces.

Normal condition. From (5.1) and (2.2), we have, for $j = 2, 3$,

$$\begin{aligned}
a^1_{2e_j+e_1} &= \tfrac{1}{3}(p_1 - p_j)^T n_j, & a^2_{2e_j+e_1} &= \tfrac{1}{3}(p'_1 - p_j)^T n_j,\\
a^1_{2e_j+e_4} &= \tfrac{1}{3}(p_4 - p_j)^T n_j, & a^2_{2e_j+e_4} &= \tfrac{1}{3}(p'_4 - p_j)^T n_j,\\
b^1_{2e_j+e_1} &= \tfrac{1}{3}(p''_1 - p_j)^T n_j, & d^1_{2e_j+e_1} &= \tfrac{1}{3}(q''_1 - p_j)^T n_j,\\
c^1_{2e_j+e_4} &= \tfrac{1}{3}(q_4 - p_j)^T n_j, & c^2_{2e_j+e_4} &= \tfrac{1}{3}(q'_4 - p_j)^T n_j.
\end{aligned} \tag{5.2}$$

C^1 *Conditions.* At present, set $a^i_{2e_4+e_j}$, $c^i_{2e_4+e_j}$, $j = 1, 2, 3, 4$, b^i_{2001}, and d^i_{2001} to any value (free parameters), and determine the other control points:

(1) *Interface of* $[p_2 p_3 p_4]$ *and* $[p_2 p_3 p'_4]$. Suppose

$$\begin{aligned}
p''_1 &= \beta^1_1 p_1 + \beta^1_2 p_2 + \beta^1_3 p_3 + \beta^1_4 p_4, & \beta^1_1 + \beta^1_2 + \beta^1_3 + \beta^1_4 &= 1,\\
p''_1 &= \beta^2_1 p'_1 + \beta^2_2 p_2 + \beta^2_3 p_3 + \beta^2_4 p'_4, & \beta^2_1 + \beta^2_2 + \beta^2_3 + \beta^2_4 &= 1.
\end{aligned} \tag{5.3}$$

Then, the C^1 conditions require (see Lemma 2.2)

$$\begin{aligned}
b^i_{1\lambda_2\lambda_3\lambda_4} = {}& \beta^i_1 a^i_{1\lambda_2\lambda_3\lambda_4} + \beta^i_2 a^i_{0\lambda_2\lambda_3\lambda_4+0100}\\
&+ \beta^i_3 a^i_{0\lambda_2\lambda_3\lambda_4+0010} + \beta^i_4 a^i_{0\lambda_2\lambda_3\lambda_4+0001}
\end{aligned} \tag{5.4}$$

for $\lambda_2\lambda_3\lambda_4 = 002, 101, 011, 110$. Hence, b^i_{1002}, b^i_{1101}, and b^i_{1011} are defined, leaving a^i_{1011} and a^i_{1101} to be determined. Equation (5.4) for $\lambda_2\lambda_3\lambda_4 = 110$ will be treated later.

(2) *Interface at* $[p_2 p_3 p''_1]$. Let

$$p''_1 = \mu_1 p_4 + \mu_2 p'_4 + \mu_3 p_2 + \mu_4 p_3, \qquad \mu_1 + \mu_2 + \mu_3 + \mu_4 = 1; \tag{5.5}$$

then C^1 conditions require

$$\begin{aligned} b^i_{\lambda_1\lambda_2\lambda_3 0+1000} = \mu_1 b^1_{\lambda_1\lambda_2\lambda_3 1} + \mu_2 b^2_{\lambda_1\lambda_2\lambda_3 1} \\ + \mu_3 b^i_{\lambda_1\lambda_2\lambda_3 0+0100} + \mu_4 b^i_{\lambda_1\lambda_2\lambda_3 0+0010} \end{aligned} \tag{5.6}$$

for $\lambda_1\lambda_2\lambda_3 = 200,\ 110,\ 101,\ 011$. Hence, b^i_{3000}, b^i_{2100}, and b^i_{2010} are defined. The equation for $\lambda_1\lambda_2\lambda_3 = 011$ will be treated later, together with (5.4).

(3) *Interface between* $[p_2p_3q_4]$, $[p_2p_3q_1'']$, *and* $[p_2p_3q_4']$. All control points of g_i' and some of the control points of f_i' can be fixed as f_i and g_i. That is, the relations (5.4)–(5.6) hold when the quantities a's, b's, β's, and μ's are substituted by c's, d's, γ's, and η's, respectively. The two untreated equations left are

$$d^i_{1110} = \gamma^i_1 a^i_{1110} + \gamma^i_2 a^i_{0210} + \gamma^i_3 a^i_{0120} + \gamma^i_4 c^i_{0111}, \tag{5.7}$$

$$d^i_{1110} = \eta_1 c^1_{0111} + \eta_2 c^2_{0111} + \eta_3 a^i_{0210} + \eta_4 a^i_{0120}, \tag{5.8}$$

where the coefficients γ_i and η_i are defined by

$$\begin{aligned} q_1'' &= \gamma^1_1 p_1 + \gamma^1_2 p_2 + \gamma^1_3 p_3 + \gamma^1_4 q_4, & \gamma^1_1 + \gamma^1_2 + \gamma^1_3 + \gamma^1_4 = 1, \\ q_1'' &= \gamma^2_1 p_1' + \gamma^2_2 p_2 + \gamma^2_3 p_3 + \gamma^2_4 q_4', & \gamma^2_1 + \gamma^2_2 + \gamma^2_3 + \gamma^2_4 = 1, \\ q_1'' &= \eta_1 q_4 + \eta_2 q_4' + \eta_3 p_2 + \eta_4 p_3, & \eta_1 + \eta_2 + \eta_3 + \eta_4 = 1. \end{aligned} \tag{5.9}$$

(4) *Interface between* $[p_1p_2p_3]$ *and* $[p_1'p_2p_3]$. Let

$$\begin{aligned} q_4 &= \alpha^1_1 p_1 + \alpha^1_2 p_2 + \alpha^1_3 p_3 + \alpha^1_4 p_4, & \alpha^1_1 + \alpha^1_2 + \alpha^1_3 + \alpha^1_4 = 1, \\ q_4' &= \alpha^2_1 p_1' + \alpha^2_2 p_2 + \alpha^2_3 p_3 + \alpha^2_4 p_4', & \alpha^2_1 + \alpha^2_2 + \alpha^2_3 + \alpha^2_4 = 1. \end{aligned} \tag{5.10}$$

Then we have

$$c^i_{0111} = \alpha^i_1 a^i_{1110} + \alpha^i_2 a^i_{0210} + \alpha^i_3 a^i_{0120} + \alpha^i_4 a^i_{0111}. \tag{5.11}$$

Now we treat the equations (5.4), (5.6), (5.7), (5.8), and (5.11). It follows from (5.4), (5.6), (5.7), and (5.8) that

$$\begin{aligned} \mu_1 a^1_{0111} + \mu_2 a^2_{0111} + \mu_3 a^i_{0210} + \mu_4 a^i_{0120} \\ = \beta^i_1 a^i_{1110} + \beta^i_2 a^i_{0210} + \beta^i_3 a^i_{0120} + \beta^i_4 a^i_{0111}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} \eta_1 c^1_{0111} + \eta_2 c^2_{0111} + \eta_3 a^i_{0210} + \eta_4 a^i_{110} \\ = \gamma^i_2 a^i_{0210} + \gamma^i_3 a^i_{0120} + \gamma^i_4 c^i_{0111}. \end{aligned} \tag{5.13}$$

Therefore, (5.11)–(5.13) form a linear system with six equations and six unknowns a^i_{0111}, a^i_{1110}, c^i_{0111}, for $i = 1, 2$. It is important to point out that this is not an independent system (see Theorem 5.1 for the solvability of the system). It has four independent equations and infinitely many solutions. In fact, if we assume that p_1, p_2, p_3, p'_1 are not coplanar and then denote

$$\begin{aligned} p_4 &= \theta_1^1 p_1 + \theta_2^1 p_2 + \theta_3^1 p_3 + \theta_4^1 p'_1, & \theta_1^1 + \theta_2^1 + \theta_3^1 + \theta_4^1 &= 1, \\ p'_4 &= \theta_1^2 p_1 + \theta_2^2 p_2 + \theta_3^2 p_3 + \theta_4^2 p'_1, & \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 &= 1, \\ q_4 &= \vartheta_1^1 p_1 + \vartheta_2^1 p_2 + \vartheta_3^1 p_3 + \vartheta_4^1 p'_1, & \vartheta_1^1 + \vartheta_2^1 + \vartheta_3^1 + \vartheta_4^1 &= 1, \\ q'_4 &= \vartheta_1^2 p_1 + \vartheta_2^2 p_2 + \vartheta_3^2 p_3 + \vartheta_4^2 p'_1, & \vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2 + \vartheta_4^2 &= 1, \end{aligned} \tag{5.14}$$

then we can derive from (5.12) and (5.13) that

$$a_{0111}^i = \theta_1^i a_{1110}^1 + \theta_2^i a_{0210}^i + \theta_3^i a_{0120}^i + \theta_4^i a_{1110}^2, \tag{5.15}$$

$$c_{0111}^i = \vartheta_1^i a_{1110}^1 + \vartheta_2^i a_{0210}^i + \vartheta_3^i a_{0120}^i + \vartheta_4^i a_{1110}^2. \tag{5.16}$$

Actually, this means any group of four weights (e.g. a_{1110}^1, a_{0210}^i *and* a_{1110}^2) defines the same 4-D hyperplane in its own barycentric coordinates (e.g., $[p_1 p_2 p_3 p'_1]$). Therefore, besides $a_0 210^1$ and $a_0 120^1$ (or $c_0 210^1$ and $c_0 120^1$, there are only 2 degrees of freedom left. We choose a_{1110}^i (or c_{1110}^i) to be the free parameters. They may be determined by approximating a quadratic (see Section 6 or Dahmen and Thamm-Schaar [1993]).

(b) *Both faces are convex.*

(b1) *Both faces are nonnegative (or nonpositive) convex.* Following the discussion of (a), the scheme for determining the control points are as before, except for the following:

(1) Only half of the control points are needed. That is, we need a_λ^i, b_λ^i for functions f_i and g_i if F and F' are nonnegative convex, or c_λ^i, d_λ^i for functions f'_i and g'_i if F and F' are nonpositive convex.
(2) a_{1110}^i (or c_{1110}^i) can be determined freely as in (a). One way to choose a_{1110}^i (or c_{1110}^i) is to make the cubic approximate a quadratic (see Section 6). In particular, $a_{1110}^i = 0$ (or $c_{1110}^i = 0$) if the face is zero convex.
(3) We now need only (5.15) for unknowns a_{0111}^1 and a_{0111}^2 if the edge $[p_2 p_3]$ is nonnegative convex, or (5.16) for unknowns c_{0111}^1 and c_{0111}^2 if the edge $[p_2 p_3]$ is nonpositive convex.

(b2) *One positive convex face and one negative convex face.* In this case, the common edge must be zero convex. Suppose F is positive convex and F' is negative convex. All of the control points are determined as before, except for the following:

(1) We only need to construct f_i, g_i, and f'_2; that is, c_λ^1 and d_λ^i are not needed. The functions g_i and f_2 have no contribution to the surface and are used for smooth transition from f_1 to f'_2.
(2) $a_{1110}^1 \le 0$ and $c_{1110}^2 \ge 0$ can be determined freely (see Section 6).

(3) We only need to have (5.11) for $i = 2$ and (5.15) for unknowns a^1_{0111}, a^2_{0111}, and c^2_{0111}.

(b3) *Both faces are zero convex.* This case, in fact, is included in case (b1). The surface is defined directly as the planar faces of the surface triangulation. No function needs to be constructed.

(c) *One convex face and one nonconvex face.* Suppose $[p_1 p_2 p_3]$ is convex and $[p'_1 p_2 p_3]$ is nonconvex, with the following exceptions:

(1) The functions f'_1 and g'_i and their control points c^1_λ and d^i_λ are not needed if F is nonnegative convex. The functions f_1 and g_i and their control points a^1_λ and b^i_λ are not needed if F is nonpositive convex.

(2) $a^1_{1110} \le 0$ (or $c^1_{1110} \ge 0$) and a^2_{1110} (c^2_{1110}) can be determined freely as in case (b). In particular, $a^1_{1110} = 0$ (or $c^1_{1110} = 0$) if $[p_1 p_2 p_3]$ is zero convex.

(3) For the treatment of equations (5.11)–(5.13), we only need to have (5.11) for $i = 2$ and (5.15) for unknowns a^1_{0111}, a^2_{0111}, and c^2_{0111} if edge $[p_2 p_3]$ is nonnegative convex, or to solve (5.11) for $i = 2$ and (5.13) for unknowns c^1_{0111}, c^2_{0111}, and a^2_{0111} if edge $[p_2 p_3]$ is nonpositive convex (see Theorem 5.1(ii) for the solvability of the system).

Coplanarity of adjacent faces. In the discussions above, we have assumed that p_1, p'_1, p_2, p_3 are affine independent. If p_1, p'_1, p_2, p_3 are coplanar, then the coefficient matrices of the linear systems (5.12) and (5.13) are singular. However, the system (5.11)–(5.13) is still solvable (see Theorem 5.1) taking a^i_{0111} or c^i_{0111} as free parameters. The other unknowns are given directly by these equations. Since the parameters a^i_{1110}, $i = 1, 2$, now become dependent, they are overly determined by the systems (5.11)–(5.13) around the three edges (e.g., $[p_1 p_2][p_2 p_3][p_3 p_1]$ for a_{i110}), and a solution may not be possible. In this case, we split the involved tetrahedron into subtetrahedra by subdividing the triangles $[p_1 p_2 p_3]$ and $[p'_1 p_2 p_3]$ into three subtriangles at their center points w and w' (a Clough–Tocher split). A solution is now possible where the coefficients are specified as before, by regarding w as p_1 and w' as p'_1.

We then need to determine the remaining coefficients over the subtetrahedra $U_1 = [p_2 p_3 p_4 w]$, $U_2 = [p_1 p_3 p_4 w]$, and $U_3 = [p_1 p_2 p_4 w]$ such that the C^1 condition is satisfied. In fact, since $w \in [p_1 p_2 p_3]$, the coefficients on the same layer are C^1 related. For the 0th layer (see Figure 9), the control points labeled • are thus already determined. The control points ∘ are determined by a coplanar condition with surrounding •. Finally, the point □ is determined from the surrounding three points ∘ by the coplanar condition.

For the 1st layer of Figure 9, the control points labeled ∘ and □ are similarly determined as the 0th layer. For the 2nd layer of Figure 9, the control points ∘ are arbitrarily chosen, and □ is determined by the coplanar condition. Finally, the 3rd-layer coefficient is free.

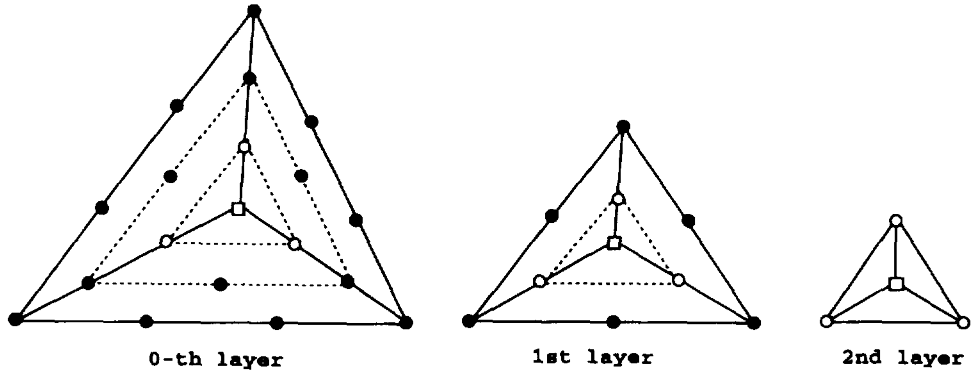


Fig. 9. Control points of zeroth, first, and second layers.

5.2 The Solvability of the Related System

Concerning the solvability of the system (5.11)–(5.13) and its subsystem, we have the following result. The proof is given in the Appendix:

THEOREM 5.1. *Given two affine independent point sets* (p_2, p_3, p'_4, p_4) *and* (p_2, p_3, q'_4, q_4), *as in Figure* 8, (*i*) *the system* (5.11)–(5.13) *has four independent equations. If* (p_1, p'_1, p_2, p_3) *is affine independent, then* (5.12) *and* (5.13) *are four independent equations for the unknowns* a^i_{0111} *and* c^i_{0111}, *for* $i = 1, 2$.

(*ii*) *Let* $\{r_1, \dots, r_6\} = \{p_1, p'_1, p_4, p'_4, q'_4, q_4\}$ *and* $\{x_1, \dots, x_6\} = \{a^1_{1110}, a^2_{1110}, a^1_{0111} a^2_{0111}, c^1_{0111}, c^2_{0111}\}$. *For any* $1 \le i < j \le 6$, *if* r_i, r_j, p_2, p_3 *are affine independent, then*

$$x_k = \phi^k_1 x_i + \phi^k_2 x_j + \phi^k_3 a^1_{0210} + \phi^k_4 a^1_{0120}, \qquad k \neq i, j, \tag{5.17}$$

where ϕ^k_i *are defined by* $r_k = \phi^k_1 r_i + \phi^k_2 r_j + \phi^k_3 p_2 + \phi^k_4 p_3$, $\phi^k_1 + \phi^k_2 + \phi^k_3 + \phi^k_4 = 1$.

5.3 Construction of Single-Sheeted A-Patches

Having built C^1 cubics with some free control points, we now illustrate how to determine these free control points such that the zero contours are three-sided or four-sided A-patches (smooth and single sheeted).

We assume (without loss of generality) that all of the normals point to the same side of the surface triangulation T. That is the side on which p_4 and p'_4 lie (see Figure 8). Under this assumption, it follows from Definition 4.1 and Eq. (5.2) that the control points on the edge, say, a^i_{0210} and a^i_{0120} on edge $[p_2 p_3]$ (see Figure 8), are nonnegative if the edge is nonnegative convex, and nonpositive if the edge is nonpositive convex. Now we can divide all of the control points into seven groups, called layers. The 0th layer consists of the control points that are "on" the faces of T. The 1st layer is next to the 0th layer, and on the same side as the normal direction, followed by the 2nd and 3rd layers. Next to the 0th layer but opposite to the normal is the $-$1st layer, and then the -2nd and -3rd layers. Now we show that we can set all of the control points on the 2nd and 3rd layer as positive and the control points on the -2nd and -3rd layers as negative.

For the face-tetrahedra, it is always possible to make the 2nd and 3rd layers' control points positive, because these control points are free under the C^0 condition. For the control points on the edge-tetrahedra, it follows from (5.4) that the 2nd and 3rd layers' control points can be positive only if the 2nd layer's control points on the neighbor face-tetrahedra are large enough. This is achieved since β_4^i in (5.4) is positive (see the proof of Proposition 5.3 for details). Similarly, the control points on the -2nd and -3rd layers can be chosen to be negative. Furthermore, all of these control points can be chosen as large as one needs in absolute value in order to get single-sheeted patches.

Since the control points around the vertices of T are determined by the normals, the smooth vertex condition is obviously satisfied. If the surface contains the edge $[p_2 p_3]$ (see Figure 8), then, since a_{1110}^i (or a_{0111}^i) is freely chosen, the smooth edge condition is easily satisfied (see the proof of Proposition 5.3). Referring to Figure 8, we prove in the following that the patches constructed over V_1 and W_1 are single sheeted. The other patches are similar:

PROPOSITION 5.2. *If the face $[p_1 p_2 p_3]$ is nonnegative convex, then the control points can be determined so that the surface over V_1 is a three-sided 4-patch.*

PROPOSITION 5.3. *If the edge $[p_2 p_3]$ is nonnegatice convex, then the control points can be determined such that the surface over W_1 is a four-sided 14-23-patch.*

Subdivision. For any face of $T = [p_1, p_2, p_3]$, if it is nonconvex and if the three inner products of the face normal and its three adjacent face normals have different signs, then subdivide the double face tetrahedra into six subtetrahedra by adding a vertex at the center w of the face (a Clough–Tocher split). The coefficients are specified as before by regarding w as p_1 (see Figure 8).

PROPOSITION 5.4. *If the above subdivision procedure is performed, then the control points can be chosen so that the surface over V_1 is a three-sided 4-patch, and the surface over W_1 is a four-sided 14-23-patch.*

A three-sided (or four-sided) patch, although by itself may be disconnected, in the case of a nonconvex face it forms a connected piece of surface with the other three-sided (or four-sided) patch of the double tetrahedra.

THEOREM 5.2. *The global piecewise surface constructed is smooth, connected, and single-sheeted.*

With Theorem 5.2, we conclude that the surface is topologically equivalent to the input triangulation.

6. SHAPE CONTROL

From the discussion in Section 5, there are several parameters that can influence the shape of the constructed C^1 surface. These parameters include

(a) the length of the normal if its orientation is fixed, (b) a^i_{1110}, and (c) $a^i_{0102} > 0$, $a^i_{1002} > 0$, $a^i_{0012} > 0$, $a^i_{0003} > 0$, and $b^i_{2001} > 0$ for $i = 1, 2$.

(a) *Interactive shape control.* The influence of the length of a normal at a vertex is that, if the normal becomes longer, then the surface becomes flatter at this point. Parameter a_{1110} lifts the surface upward to the top vertex of the tetrahedron, while others push the surface downward toward the bottom of the tetrahedron. In order to get a desirable surface, one may specify some additional data points in the tetrahedron considered and then approximate these points in the least-squares sense.

(b) *Default shape control.* Here we only consider the effect of the free parameters; that is, suppose the normals are fixed. The aim of the default choice of these parameters is to avoid producing bumpy surfaces. The commonly used method is to keep the surface patch close to a quadric patch [Bajaj 1992; Dahmen and Thamm-Schaar 1993].

By least-squares approximation of the coefficients of a quadric [Dahmen and Thamm-Shaar 1993], one can derive that

$$a_{1110} = \frac{1}{4}(a_{1200} + a_{2100} + a_{2010} + a_{1020} + a_{0210} + a_{0120}).$$

Using the same idea, the other parameters can also be determined. For example, a_λ for $\lambda_4 > 1$ can be determined by the degree elevation formula

$$a_\lambda = \frac{1}{3}\sum_{i=1}^{4} \lambda_i x_{\lambda - e_i}, \qquad |\lambda| = 3, \quad \lambda_4 > 1, \tag{6.1}$$

where $x_{\lambda - e_i}$ is the solution of the following equations in the least-squares sense:

$$a_\lambda = \frac{1}{3}\sum_{i=1}^{4} \lambda_i x_{\lambda - e_i}, \qquad |\lambda| = 3, \quad \lambda_4 = 0, 1.$$

In the same way, b_{2001} can be determined. Therefore, under the C^1 conditions, we can define two sets of control points $\{a^s_\lambda\}$ and $\{a^q_\lambda\}$ over V_1, where $\{a^s_\lambda\}$ is yielded from the single-sheeted consideration (see Propositions 5.2–5.4) and where $\{a^q_\lambda\}$ comes from approximating a simple (quadratic) surface. Note that the surface defined by $\{a^s_\lambda\}$ above may not be desirable in shape, while the surface defined by $\{a^q_\lambda\}$ above may not be single sheeted. In our implementation we take a finite sequence $0 = t_0 < t_1 < \cdots < t_m = 1$ and consider $\{a^{(i)}_\lambda\} = \{(1 - t_i)a^q_\lambda + t_i a^s_\lambda\}$, $i = 0, 1, \ldots, m$, selecting the single-sheeted surface defined by $\{a^{(i)}_\lambda\}$ for the smallest index i. Experiments show that this approach works well and that a desirable surface is obtained with $t_i < 0.5$. Examples are shown later in Figure 11.

7. EXAMPLES

Examples of the simplicial hull construction and C^1 smoothed triangulations using cubic A-patches are shown in Figures 10–13. Color pictures Figures 12 and 13 are also provided at the end of the paper. Note in these figures how the "convex" faces are smoothed by a single cubic A-patch per face, while a

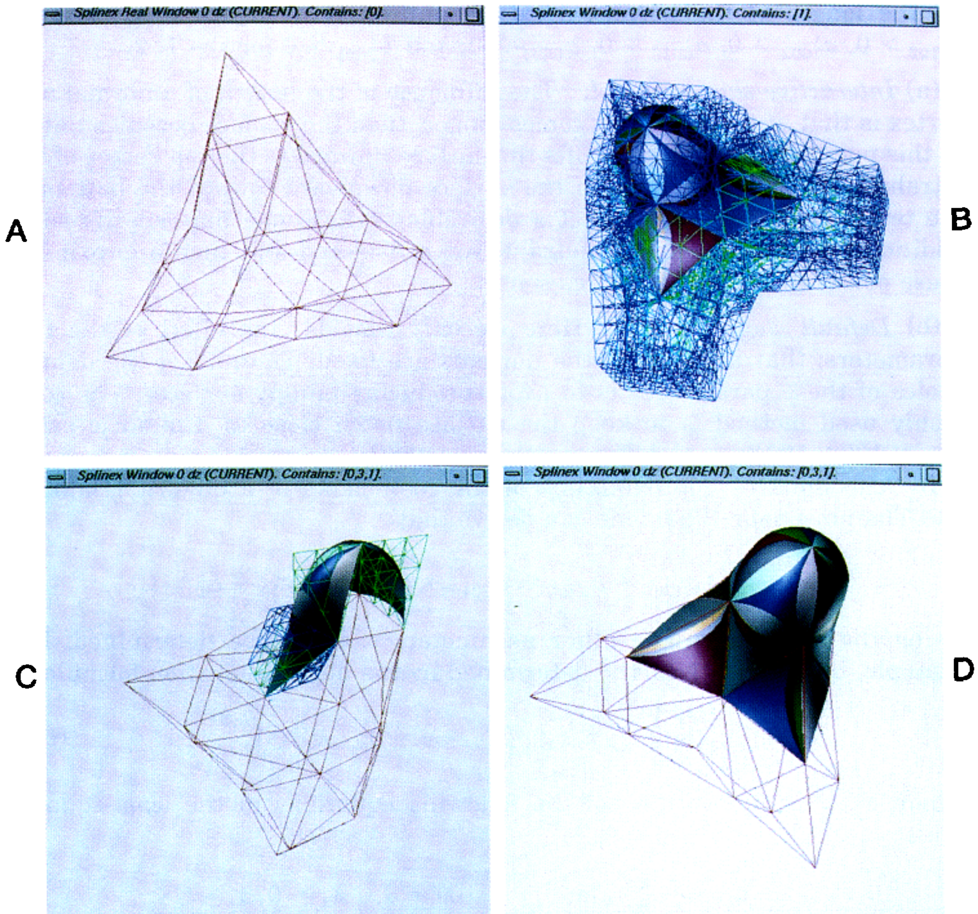


Fig. 10. Surface triangulation, the simplicial hull, and some of the interpolatory C^1 cubic A-patches.

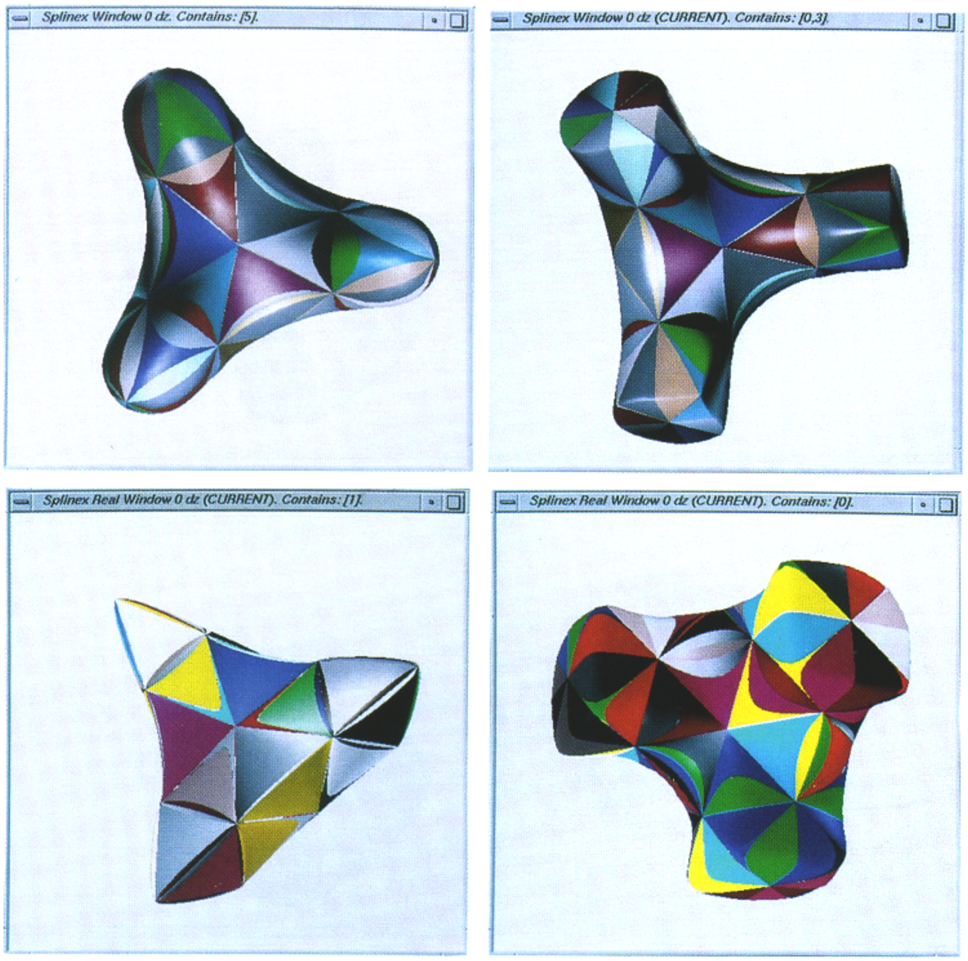
Clough–Tocher splitting occurs for coplanar faces and some "nonconvex" faces, as determined by the vertex normals assignment and the adjacent faces.

APPENDIX

PROOF OF LEMMA 3.1. Let $g(\alpha_1, \alpha_2, \alpha_3) = F(\alpha_1, \alpha_2, \alpha_3, 1 - \alpha_1 - \alpha_2 - \alpha_3)$. The smoothness of the surface patch S_F requires that $\nabla g(\alpha_1, \alpha_2, \alpha_3) \neq 0$ for every $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ on S_F. We prove only the smoothness of the three-sided j-patch. The proof of smoothness of the four-sided patch is similar.

Suppose the three-sided j-patch is not smooth. There will then be a point $\alpha^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*)^T \in S_F$ in the interior of S such that $\nabla g = 0$. Since

$$\frac{\partial g}{\partial \alpha_i} = \frac{\partial F}{\partial \alpha_i} - \frac{\partial F}{\partial \alpha_4}, \qquad i = 1, 2, 3,$$



A B

C D

Fig. 11. Different smoothings of the surface triangulation using C^1 cubic A-patches

we have

$$\frac{\partial F}{\partial \alpha_1} = \frac{\partial F}{\partial \alpha_2} = \cdots = \frac{\partial F}{\partial \alpha_4}.$$

Using Euler's formula [Walker 1978] for homogeneous polynomials

$$\sum_{i=1}^{4} \alpha_i \frac{\partial F}{\partial \alpha_i} = 4F \qquad \text{and} \qquad \sum_{i=1}^{4} \alpha_i = 1,$$

we have $\partial F/(\partial \alpha_i) = 0$, $i = 1, \ldots, 4$. Let $p_1 \in S_j$ and $t = t^* \in (0, 1)$ such that $\alpha^* = t^* e_i + (1 - t^*) p_1 = \alpha(t^*)$; that is, $F(\alpha(t^*)) = 0$. Furthermore, let

$$\left.\frac{\partial F(\alpha(t))}{\partial t}\right|_{t=t^*} = \sum_{i=1}^{4} \frac{\partial F}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial t} = 0.$$

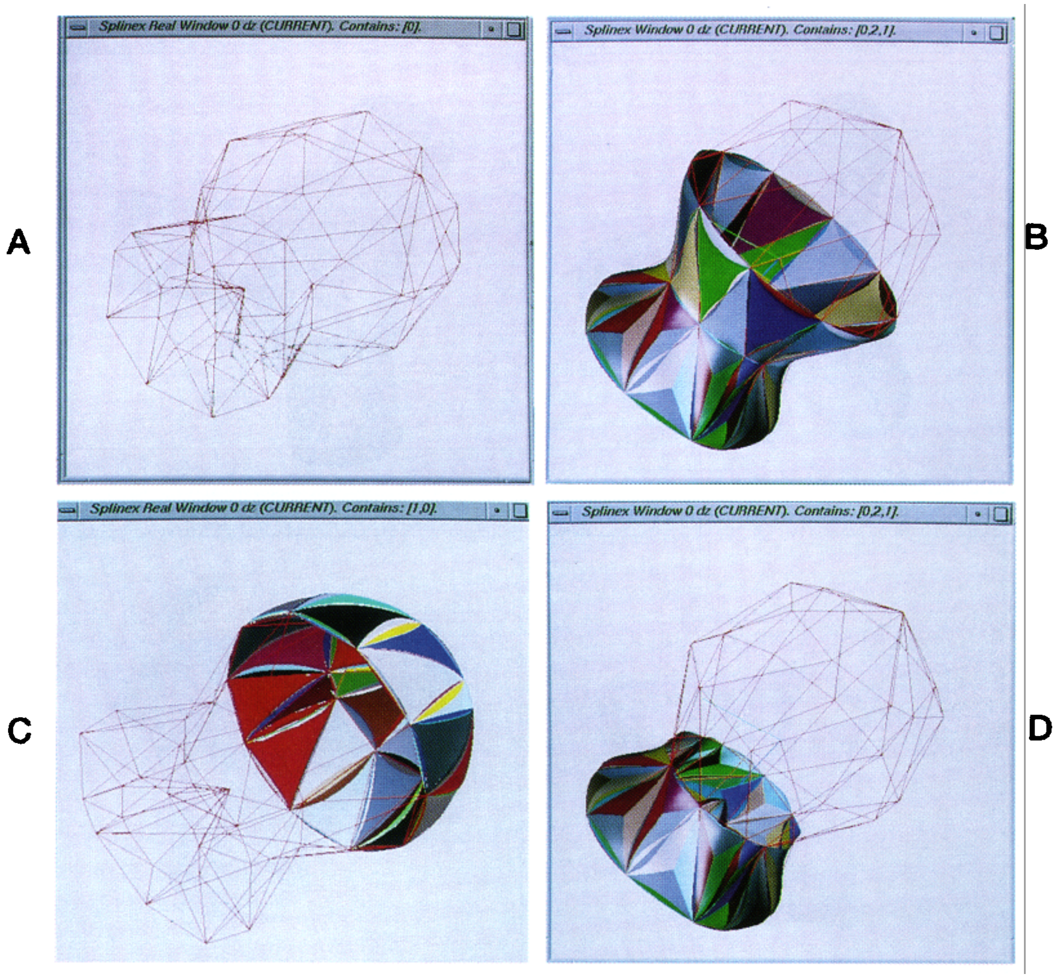


Fig. 12. Surface triangulation and some of the interpolatory C^1 cubic A-patches.

This implies that t^* is a double zero of $F(\alpha(t))$, a contradiction to the definition of the three-sided patch. □

Proof of Theorem 3.2. For the sake of simplicity, we assume that $j = 4$. Let $p = (y_1, y_2, y_3, 0)^T \in S_4$ (i.e., $y_i > 0, \sum_{i=1}^{3} y_i = 1$),

$$\alpha(t) = te_4 + (1 - t)p = ((1 - t)y_1, (1 - t)y_2, (1 - t)y_3, t)^T,$$

for $t \in (0, 1)$. Then

$$\begin{aligned} F(\alpha(t)) &= \sum_{|\lambda| = n} \frac{b_\lambda n!}{\lambda!} y_1^{\lambda_1} y_2^{\lambda_2} y_3^{\lambda_3} (1 - t)^{\lambda_1 + \lambda_2 + \lambda_3} t^{\lambda_4} \\ &= \sum_{|\lambda| = n} \frac{b_\lambda (\lambda_1 + \lambda_2 + \lambda_3)!}{\lambda_1! \lambda_2! \lambda_3!} y_1^{\lambda_1} y_2^{\lambda_2} y_3^{\lambda_3} \end{aligned}$$

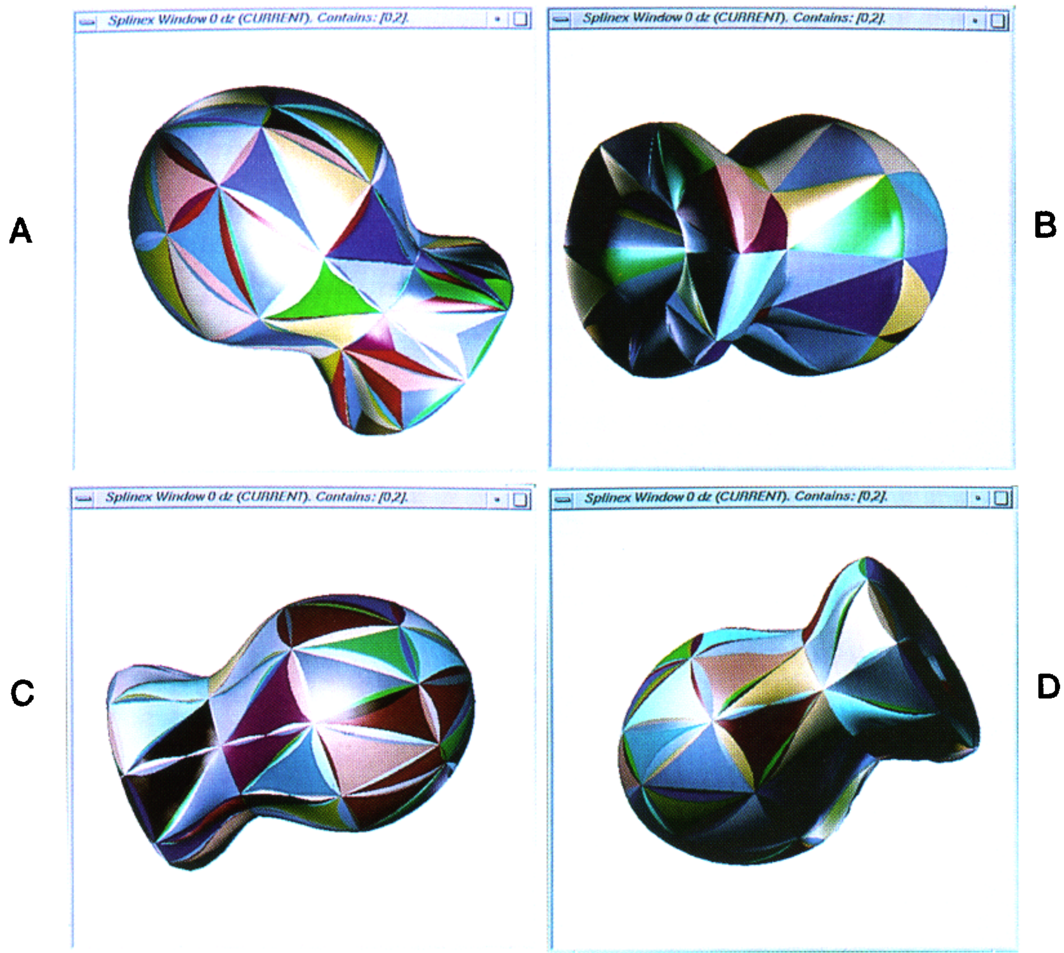


Fig. 13. Complete smoothing of the surface triangulation using C^1 cubic A-patches

$$\times \frac{n!}{(\lambda_1 + \lambda_2 + \lambda_3)!\lambda_4!}(1 - t)^{\lambda_1 + \lambda_2 + \lambda_3} t^{\lambda_4} \tag{A.1}$$

$$= \sum_{l=0}^{n} \left(\sum_{\substack{|\lambda| = n \\ \lambda_4 = l}} b_\lambda B^{n-l}_{\lambda_1\lambda_2\lambda_3}(y_1, y_2, y_3) \right) B_l^n(t)$$

$$= \sum_{l=0}^{n} B_l(y_1, y_2, y_3) B_l^n(t).$$

By (3.1) and (3.2), $B_0 > 0$ if $k > 0$, $B_l \geq 0$, for $l = 1, \ldots, k - 1$, and $B_l \leq 0$, for $l = k + 1, \ldots, n$. If $B_n = \cdots = B_{n-m-1} = 0$ and $B_{n-m} < 0$ for some m with $0 \leq m \leq n - k - 1$, then $F(\alpha(t))$ can be written as

$$F(\alpha(t)) = (1 - t)^m \sum_{l=0}^{n-m} C_l(y_1, y_2, y_3) B_l^{n-m}(t), \tag{A.2}$$

where $C_0 > 0$ if $k > 0$, $C_{n-m} < 0$; and the sequence $C_0, C_1 \cdots C_{n-m}$ has at most one sign change. By the variation diminishing property of the functional BB form, the equation $F(\alpha(t))$ has at most one root in (0, 1). Finally, we need to show that the surface at the boundary of the tetrahedron is smooth. In the proof above, if we allow the intersection to occur at the boundary, then there may be an intersection of higher multiplicity at $t = 0$ or $t = 1$. That is, the surface contains vertices or edges of the tetrahedron. Here, the smooth vertex and smooth edge conditions in the theorem guarantee that the surface is also smooth on the boundary of S. □

PROOF OF THEOREM 3.3. Without loss of generality, we assume that $(i, j, k, l) = (1, 2, 3, 4)$. Then the edge $[e_1e_2]$ and $[e_3e_4]$ can be expressed as

$$[e_1e_2] = \{p : p = ue_1 + (1-u)e_2, u \in [0,1]\},$$
$$[e_3e_4] = \{p : p = ve_3 + (1-v)e_4, v \in [0,1]\},$$

and the line segment passing through the two edges is

$$\alpha(t) = t[e_1e_2] + (1-t)[e_3e_4] = (ut, (1-u)t, v(1-t), (1-v)(1-t))^T,$$

for $t \in (0, 1)$. Hence,

$$\begin{aligned}
F(\alpha(t)) &= \sum_{|\lambda|=n} \frac{b_\lambda n!}{\lambda!} u^{\lambda_1}(1-u)^{\lambda_2} v^{\lambda_3}(1-v)^{\lambda_4} t^{\lambda_1+\lambda_2}(1-t)^{\lambda_3+\lambda_4} \\
&= \sum_{l=0}^{n} \left(\sum_{\substack{|\lambda|=n \\ \lambda_1+\lambda_2=l}} \frac{b_\lambda(\lambda_1+\lambda_2)!(\lambda_3+\lambda_4)!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!} u^{\lambda_1}(1-u)^{\lambda_2} v^{\lambda_3}(1-v)^{\lambda_4} \right) \\
&\quad \times \frac{n!}{l!(n-l)!} t^l (1-t)^{n-l} \\
&= \sum_{l=0}^{n} B_l(u,v) B_l^n(t).
\end{aligned}$$

It follows from (3.3) and (3.4) that $F(\alpha(t))$ has at most one zero in (0, 1). Again, the smooth vertex and smooth edge conditions in the theorem guarantee that the surface is smooth on the boundary of S. □

PROOFS OF THE PROPERTIES OF A-PATCHES. Property (a) can be verified by reconsidering the proof of Theorem 3.2. For example, if $m = 1$, $j = 4$, $(y_1, y_2, y_3) = (1, 0, 0)$ at e_m. Hence,

$$B_l(y_1, y_2, y_3) = b_{(n-l)e_1 + le_4} = 0, \qquad l = 0, 1, \ldots, k.$$

Therefore, $t = 0$ is the root of $F(\alpha(t))$ with multiplicity $k + 1$. On the other hand, e_m is not a singular point of S_F, since $b_\lambda \neq 0$ for $\lambda = (n-1)e_m + e_s$.
□

We illustrate property (b) by showing that any line passing through edge $[e_i e_j]$ and vertex e_k is tangent to S_F with multiplicity s. In fact, if we take $v = 1$ in the proof of Theorem 3.3, we have

$$B_l(u, v) = B_l(u, 1) = b_{\lambda_i e_i + \lambda_j e_j + (n-l)e_k} = 0.$$

Hence, $t = 0$ is a root of $F(\alpha(t))$ with multiplicity $s + 1$. Again, e_k is not a singular point. □

The proof of property (c) is similar to the proof of property (a).

For Property (d), the mappings are given by the definition. A point $P \in S_F$ maps to $\alpha^* \in S_j$ if and only if line segment (e_j, α^*) intersects S_F at P. And the mapping is one to one for $\alpha^* \in S'_j = \{\alpha \in S_j, F(e_j) \cdot F(\alpha^*) \geq 0\}$, as (e_j, α^*) intersects S_F an odd number times iff $F(e_j) \cdot F(\alpha^*) \geq 0$.

The proof of Property (e) is similar to (d).

Proof of Theorem 5.1. (i) The system (5.11)–(5.13) can be written as $XA = -[a^i_{0210} a^i_{0120}]B$, where

$$A = \begin{bmatrix} \alpha_1^1 & 0 & \beta_1^1 & 0 & \gamma_1^1 & 0 \\ 0 & \alpha_1^2 & 0 & \beta_1^2 & 0 & \gamma_1^2 \\ \alpha_4^1 & 0 & \beta_4^1 - \mu_1 & -\mu_1 & 0 & 0 \\ 0 & \alpha_4^2 & -\mu_2 & \beta_4^2 - \mu_2 & 0 & 0 \\ -1 & 0 & 0 & 0 & \gamma_4^1 - \eta_1 & -\eta_1 \\ 0 & -1 & 0 & 0 & -\eta_2 & \gamma_4^2 - \eta^2 \end{bmatrix},$$

$$B = \begin{bmatrix} \alpha_2^1 & \alpha_2^2 & \beta_2^1 - \mu_3 & \beta_2^2 - \mu_3 & \gamma_2^1 - \eta_3 & \gamma_2^2 - \eta_3 \\ \alpha_3^1 & \alpha_3^2 & \beta_3^1 - \mu_4 & \beta_3^2 - \mu_4 & \gamma_3^1 - \eta_4 & \gamma_3^2 - \eta_4 \end{bmatrix}.$$

It follows from (5.3), (5.5), (5.9), and (5.10) that

$$\begin{bmatrix} p_1 & p'_1 & p_4 & p'_4 & q_4 & q'_4 & p_2 & p_3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0.$$

Hence, the rank of the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is at most four; that is, the matrix A is singular. Since $\beta_1^1 \neq 0$, $\beta_1^2 \neq 0$, the first two rows and the last two rows of A are independent. That is, matrix A has rank four. Hence, the system (5.11)–(5.13) has four independent equations. Now we show that, if (p_1, p'_1, p_2, p_3) is affine independent, then the submatrices A_1 and A_2 are nonsingular, where

$$A_1 = \begin{bmatrix} \beta_4^1 - \mu_1 & -\mu_1 \\ -\mu_2 & \beta_4^2 - \mu_2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \gamma_4^1 - \eta_1 & -\eta_1 \\ -\eta_2 & \gamma_4^2 - \eta_2 \end{bmatrix}$$

are the submatrices of A and are the coefficient matrices of the equations (5.12) and (5.13), respectively. This implies that (5.12) and (5.13) are four independent equations for unknowns a^i_{0111} and c^i_{0111}, $i = 1, 2$. In fact, the affine independency of (p_1, p'_1, p_2, p_3) is the necessary and sufficient condition for the nonsingularity of A_1 and A_2. It follows from (5.3) and (5.5) that

$$\begin{bmatrix} p_4 & p'_4 \\ 1 & 1 \end{bmatrix} A_1 = -\begin{bmatrix} p_1 & p'_1 & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} B_1,$$
$$B_1 = \begin{bmatrix} \beta^1_1 & 0 \\ 0 & \beta^2_1 \\ \beta^1_2 - \mu_3 & \beta^2_2 - \mu_3 \\ \beta^1_3 - \mu_4 & \beta^2_3 - \mu_4 \end{bmatrix}. \tag{A.3}$$

Since $p_4 \neq p'_4$ and $\beta^1_1 \neq 0$, $\beta^2_1 \neq 0$, matrix A_1 is of full rank if the matrix $\begin{bmatrix} p_1 & p'_1 & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is nonsingular. On the other hand, if this matrix is singular, that is, if p_1, p'_1, p_2, p_3 are coplanar, then the matrix A_1 is also singular. Otherwise, p_4, p'_4 will lie on the plane $\langle p_1 p'_1 p_2 p_3 \rangle$ by (A.3), which yields a contradiction. Similarly, A_2 is nonsingular iff $\begin{bmatrix} p_1 & p'_1 & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is nonsingular.

(ii a) If p_1, p'_1, p_2, p_3 are affine independent, then by (A.3) we know that a^i_{0111}, $i = 1, 2$, can be expressed is an affine combination of a^i_{1110} and a^1_{0210}, a^1_{0120}. By (5.11), c^i_{0111}, $i = 1, 2$ can also be expressed as an affine combination of a^i_{1110} and a^1_{0210}, a^1_{0120}.

(b) If we take p_4, p'_4, p_2, p_3 or q_4, q'_4, p_2, p_3 to be the affine independent set, then the equations (5.11)–(5.13) are already in the form (5.17).

(c) Any other cases can be derived from one of the above cases. □

PROOF OF PROPOSITION 5.2. Since the 0th layer's control points are nonpositive, and the 2nd and 3rd layers' control points can be set as positive, the defined surface is then a three-sided 4-patch (see Theorem 3.2). □

PROOF OF PROPOSITION 5.3. First, the 0th layer's control points are nonpositive. Now we show that the 2nd and 3rd layers' control points can be set as positive. Since p'' is above the planes $\langle p_1 p_2 p_3 \rangle$ and $\langle p'_1 p_2 p_3 \rangle$ (i.e., it is at the same side as p_4 of the planes), then $\beta^i_4 > 0$. Hence, from (5.4) and (5.5), b_{1101} and b_{1011} can be set as positive if a_{0102} and a_{0012} are chosen large enough. Similarly, by (5.3), $b_{1002} > 0$ if a_{0003} is chosen large enough. Also, b_{2001} can be set as positive since it is free. Now it follows from (5.8)–(5.10) and $\mu_1 > 0$, $\mu_2 > 0$ that b_{2100}, b_{2010}, and b_{3000} can be set as positive. Therefore, the surface defined in this way is a four-sided 14-23-patch over W_1 if $[p_2 p_3]$ is positive convex. If $[p_2 p_3]$ is zero convex, that is, if $a^i_{0210} = a^i_{0120} = 0$, then by (5.12) we can make $a^i_{0111} > 0$ and $b^i_{1110} > 0$ by choosing the free parameter a^i_{1110}, $i = 1, 2$. Hence, the 1st layer's control points are nonnegative. Hence, here the patch over W_1 degenerates to the edge $[p_2 p_3]$, and the smooth edges condition is satisfied. However, if the parameter a^i_{1110} is overdetermined, then a subdivision as in the coplanar case is needed. □

PROOF OF PROPOSITION 5.4. (i) $[p'_1 p_2 p_3]$ is a nonconvex face (see Figure 8). We show that all of the 1st layer's control points over V_i and W_i, $i = 1, 2$, can be set as nonpositive, and the -1st layer's control points over V'_i, $i = 1, 2$, and W'_i, $i = 1, 2$, can be set as nonpositive. If p_1, p'_1, p_2, p_3 are affine independent, then we use the equalities (5.15) and (5.16). Since both p_4 and p'_4 are at the same side of the surface triangulation T, $\theta^i_1 \theta^i_4 > 0$ for $i = 1, 2$. Assume, without loss of generality, that $\theta^1_1 > 0$, $\theta^1_4 > 0$ and $\theta^2_1 > 0$, $\theta^2_4 > 0$, so $\vartheta^1_1 < 0$, $\vartheta^1_4 < 0$ and $\vartheta^2_1 < 0$, $\vartheta^2_4 < 0$. Then, by (5.15) and (5.16), we can take a^i_{1110} large enough such that $a^i_{0111} > 0$ and $c^i_{0111} < 0$. Furthermore, their absolute value can be larger than any specified value. Since the 1st and -1st layers' control points that are determined by the normals are nonnegative and nonpositive, respectively, then all of the 1st and -1st layers' control points can be set as nonnegative and nonpositive, respectively. Therefore, the surfaces over V_i and V'_i are three-sided 4-patches, and the surfaces over W_i and W'_i are four-sided 14-23-patches (see Theorem 3.2 and 3.3). If p_1, p'_1, p_2, p_3 are coplanar (not affine independent), then by Theorem 5.1, all of the unknowns can be expressed linearly by a^i_{0111}, $i = 1, 2$ (or c^i_{0111}, $i = 1, 2$). It is easy to see that we can take $a^i_{0111} > 0$ (or $c^i_{0111} < 0$) large (or small) enough so that $c^i_{0111} < 0$ (or $a^i_{0111} > 0$).

(ii) If $[p'_1 p_2 p_3]$ is convex, then the edge $[p_2 p_3]$ is also convex. Then Propositions 5.2 and 5.3 can be used for this face and edge. As for the face $[p_1 p_2 p_3]$, the discussion above can be used.

Finally, we point out why the splitting is necessary. Consider the face $[p_1 p_2 p_3]$ as an example (see Figure 8). In order to have a^1_{0111}, a^1_{1101}, a^1_{1011} greater than zero, a^1_{1110} has to be determined three times by the three C^1 constraints if no splitting is performed. Therefore, in general, a solution is impossible without splitting. Also note that if the three inner products between the face normal and its neighbor's face normals have the same sign (positive or negative), then a^1_{1110} can be determined so that a^1_{0111}, a^1_{1101}, a^1_{1011} are greater than zero. Hence, we do not need to split the face here. □

PROOF OF THEOREM 5.2. We consider as the general case double face tetrahedra and double edge tetrahedra.

(*Smoothness*). The whole surface is smooth as each single piece is smooth.

(*Connectedness and single-sheetedness*). Let $S = [p_1 p_2 p_3 p_4]$ and $S' = [p_1 p_2 p_3 q_4]$ be two-face tetrahedra sharing face S_4, S_F and S'_F, be the two three-sided 4-patches over them. From the single-sheeted construction, $F(e_4) < 0$ and $F'(e_4) > 0$ and from the C_0 conditions, for $\alpha^* \in S_4$, "double" patch $D = S_F \wedge S'_F$, exactly once. Hence D is single-sheeted over the double tetrahedra. In particular, $p_4 \alpha^* q_4$ intersects S'_F, when $F(\alpha^*) > 0$, S_F when $F(\alpha^*) < 0$, and both S_F and $S'_{F'}$, at α^* when $F(\alpha^*) = 0$ where S_F and $S'_{F'}$, meet.

Regard the "double" surface D as a function of α^* denoted as $D(\alpha^*)$. From the fact that the two patches are smooth at least C^0 to each other,

$$\lim_{\theta^* \to \alpha^*,\, \theta \in S_4} D(\theta^*) = D(\alpha^*).$$

Hence D is connected over the double tetrahedra $S \wedge S'$.

Similarly, let $S = [p'_1 p_2 p_3 p_4]$ and $S' = [q'_1 p_2 p_3 q_4]$ be two edge tetrahedra sharing edge $[p_2 p_3]$, and S_F, $S'_{F'}$, be the two four-sided 14-23-patches over them. From the single-sheeted construction, $F(\beta^*) < 0$ for $\beta^* \in [p'_1 p_4]$ and $F'(\gamma^*) > 0$ for $\gamma^* \in [q'_1 q_4]$. From the C^0 conditions, $F(\alpha^*) = F'(\alpha^*)$, for $\alpha^* \in [p_2 p_3]$. Again from Property (e), polyline $\beta^* \alpha^* \gamma^*$ intersects the double patch $D = S_F \wedge S'_{F'}$ exactly once. Hence D is single-sheeted over the double tetrahedra. In particular, $\beta^* \alpha^* \gamma^*$ intersects $S'_{F'}$ when $F(\alpha^*) > 0$, S_F when $F(\alpha^*) < 0$, and both S_F and $S'_{F'}$, at α when $F(\alpha^*) = 0$ where S_F and $S'_{F'}$ meet. Regard D as a function of (α, β), denoted as $D(\alpha, \beta)$, where $(\alpha, 1 - \alpha)$ is the barycentric coordinate of α^* and $(\beta, 1 - \beta)$ is the barycentric coordinate of β^* or γ^*. From the fact that the two patches are smooth and at least C^0 to each other, we have,

$$\lim_{\theta \to \alpha,\, \phi \to \beta} D(\theta, \phi) = D(\alpha, \beta).$$

Hence D is connected over the double tetrahedra $S \wedge S'$. □

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