Piecewise Rational Approximations of Real Algebraic Curves

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Abstract

We use a combination of both algebraic and numerical techniques to construct a C^1-continuous, piecewise (m, n) rational ϵ-approximation of a real algebraic plane curve of degree d. At singular points we use the classical Weierstrass Preparation Theorem and Newton power series factorizations, based on the technique of Hensel lifting. These, together with modified rational Padé approximations, are used to efficiently construct locally approximate, rational parametric representations for all real branches of an algebraic plane curve. Besides singular points we obtain an adaptive selection of simple points about which the curve approximations yield a small number of pieces yet achieve C^1 continuity between pieces. The simpler cases of C^{-1} and C^0 continuity are also handled in a similar manner. The computation of singularity, the approximation error bounds and details of the implementation of these algorithms are also provided.

1 Introduction

An algebraic plane curve $\mathbf{C}$ of degree d in $\mathbb{R}^2$ is implicitly defined by a single polynomial equation $f(x, y) = 0$ of degree d with coefficients in $\mathbb{R}$. A rational algebraic curve of degree d in $\mathbb{R}^2$ can additionally be defined by rational parametric equations which are given as $(x = G_1(u), y = G_2(u))$, where G_1 and G_2 are rational functions in u of degree d, i.e., each is a quotient of polynomials in u of maximum degree d with coefficients in $\mathbb{R}$. Rational curves are only a subset of implicit algebraic curves of degree $d+1$. While all degree two curves (conics) are rational, only a subset of degree three (cubics) and higher degree curves are rational. In general, a necessary and sufficient condition for the global rationality of an algebraic curve of arbitrary degree is given by the Cayley-Riemann criterion: a curve is rational if and only if $g = 0$, where g, the genus of the curve is a measure of the deficiency of the curve's singularities from its maximum allowable limit [1, 23].

The Rational Approximation Problem

Given a real algebraic plane curve $\mathbf{C}$: $f(x, y) = 0$ of degree d and of arbitrary genus, a box B defined by $\{(x, y) | \alpha \le x \le \beta, \gamma \le y \le \delta\}$, an error bound $\epsilon > 0$, and integers m, n with $m + n \le d$ construct a C^{-1}, C^0 or C^1 continuous piecewise rational ϵ-approximation of

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all portions of $\mathbf{C}$ within the given bounding box B, with each rational function $\frac{P_i}{Q_i}$ of degree $P_i \leq m$ and degree $Q_i \leq n$. Here C^{-1} means no continuity condition is imposed between the different pieces, C^0 implies there are no gaps and C^1 implies that the first derivatives are continuous at the common end points of adjacent pieces. The ϵ-approximation here means that the approximation error is within given ϵ. The input curve $f(x,y) = 0$ may be reducible and have several real components but we assume it has components of only single multiplicity i.e. polynomial $f(x,y)$ has no repeated factors.

Results

We use a combination of both algebraic and numerical techniques to construct a C^1-continuous, piecewise (m,n) rational ϵ-approximation by two different approaches, of a real algebraic plane curve. At singular points we rely on the classical resolution of plane curves [1, 23] based on the Weierstrass Preparation Theorem [24] and Newton power series factorizations[17], using the technique of Hensel lifting[14]. These, together with modified Padé approximations, are used to efficiently construct locally approximate, rational parametric representations for all real branches of an algebraic plane curve. Besides singular points we obtain an adaptive selection of simple points about which the curve approximations yield a small number of pieces yet achieve C^1 continuity between pieces. The simpler cases of C^{-1} and C^0 continuity are also handled in a similar manner. The computation of singularity, the approximation error bounds and details of the implementation of these algorithms are also provided.

Applications

In geometric design and computer graphics one often uses rational algebraic curves and surfaces because of the advantages obtained from having both the implicit and rational parametric representations [4], [20]. While the rational parametric form of representing a curve allows efficient tracings, ease for transformations and shape control, the implicit form is preferred for testing whether a point is on the given curve, is on the left or right of the curve and is further conducive to the direct application of algebraic techniques. Simpler algorithms are also possible when both representations are available. For example, a straightforward method exists for computing curve - curve and surface - surface intersection approximations when one of the curves, respectively surfaces, is in its implicit form and the other in its parametric form. Global parameterization algorithms exist for implicit algebraic curves of genus zero [2, 3] which allows one to compute this dual representation. A solution to our rational approximation problem yields a rational representation, although approximate, and with all the above advantages for arbitrary genus algebraic plane curves. Perhaps even more important, there are requirements to approximate the algebraic curves in a computer aided geometric design environment. A contour of an algebraic surface, even in its functional form $z = f(x,y)$ is an algebraic curve. The offset of an algebraic curve, even its parametric form, is an algebraic curve either.

Prior Work

In [7, 16], power series are constructed to locally approximate plane algebraic curves and surface intersections at simple points. The method of [16] technically relies on the Implicit Function Theorem, seeking to represent a curve branch explicitly in one coordinate as function of the other coordinate(s), while [7] uses a Taylor series expansion. Both these methods however do not seem to have a natural extension that handles singular points. Papers [7] and [15] survey a number of techniques for generating a piecewise linear approximation of an algebraic curve. Further, [21, 22] present techniques for parametric curve approximations which work only for special cases and simple singularities. In this paper we extend these to higher order rational function approximations of the curve as well as deal with all real singularities of the given algebraic curve. Experiences shows that the (m,n) rational functions are often better approximation than degree $m+n$ polynomials to an algebraic curve, and further (m,n) rational functions are much lower degree algebraic curves than degree $m+n$ polynomials.

Methods for computing local branch parameterizations at singular points have been presented in [10, 11, 13], based on the Newton polygon, see [23]. We instead use the iterative lifting technique of Hensel coupled with a univariate Padé approximation algorithm, both based on the fast euclidean HGCD method of [8].

Given a sampling of points (x_i, y_i) on the plane curve $\mathbf{C}$: $f(x,y) = 0$ of degree d, an alternate powerful approximation scheme is is to use B-splines to generate piecewise parametric polynomial or rational approximations with C^{k-1} continuity using degree $k < d$ parametric polynomials [19, 12, 18]. We show here that C^1 Padé rational approximations, with controllable approximation error, can also be easily constructed since it is based on the same GCD algorithm as the one required for the curve resolution at singular points. Sampling points on a curve $f(x,y) = 0$ in a correct order is to trace the curve by piecewise approximation[7]. A better approximation of the sample points need more points. Hence need more approximation pieces during tracing procedure. After the sampling, the fitting procedure commonly used does not use the information of $f(x,y) = 0$. It is then hard to control the error of the fitting to the original curve. Our approach combine the sampling and fitting into one step with controllable error. Furthermore, the rational approximation, which is in parametric form, in our approach can of course be easily sampled to get the discrete points if necessary.

2 Sketch of Algorithm

Input Given a real algebraic curve $\mathbf{C}$ of degree d, a bounding box B, a finite precision real number ϵ and integers m, n with $m + n \leq d$.

Output A C^{-1}, C^0 or C^1 continuous piecewise rational ϵ-approximation of all portions of $\mathbf{C}$ within the given bounding box B, with each rational function $\frac{P_i}{Q_i}$ of degree $P_i \leq m$ and degree $Q_i \leq n$ and $m + n \leq d$.

Algorithm We state the algorithm for a C^1 continuous piecewise rational ϵ-approximation. The C^{-1} and C^0 are similar and simpler.

1. Compute all intersections of the given real plane curve $\mathbf{C}$ within the given bounding box B and also the tracing direction at these points. Let the curve within the box B be denoted by $\mathbf{C_B}$. Next, compute all singular points S and x-extreme points T on the bounded plane curve $\mathbf{C_B}$. The set of points T act as starting points for smooth ovals of the curve $\mathbf{C}$ completely inside B.

2. Compute a Newton factorization for each singular point (x_i, y_i) in S and obtain a power series representation for each analytic branch of $\mathbf{C}$ at (x_i, y_i) and given by

$$\begin{cases} X(s) = x_i + s^{k_i} \\ Y(s) = \sum_{j=0}^{\infty} c_j^{(i)} s^j, \quad c_0^{(i)} = y_i \end{cases} \tag{2.1}$$

or

$$\begin{cases} Y(s) = y_i + s^{k_i} \\ X(s) = \sum_{j=0}^{\infty} \tilde{c}_j^{(i)} s^j, \quad \tilde{c}_0^{(i)} = x_i \end{cases} \tag{2.2}$$

3. Without loss of generality, we only consider the case where the analytic branch at the singularity is of type (2.1). Compute $\frac{P_{mn}(s)}{Q_{mn}(s)}$ the (m,n) Padé approximation of $Y(s)$. That is $\frac{P_{mn}(s)}{Q_{mn}(s)} - Y(s) = O(s^{m+n+1})$

4. Compute $\beta > 0$ a real number, corresponding to points $(\tilde{x}_i = X(\beta), \tilde{y}_i = Y(\beta))$ and $(\hat{x}_i = X(-\beta), \hat{y}_i = Y(-\beta))$ on the analytic branch of the original curve $\mathbf{C}$, such that $\frac{P_{mn}(s)}{Q_{mn}(s)}$ is convergent for $s \in [-\beta, \beta]$ (see sub-sections (3.3.2) and (3.3.4) for details).

5. Modify $P_{mn}(s)/Q_{mn}(s)$ to $\tilde{P}_{mn}(s)/\tilde{Q}_{mn}(s)$ such that $\tilde{P}_{mn}(s)/\tilde{Q}_{mn}(s)$ is C^1 continuous approximation of $Y(s)$ on $[0, \beta]$, similarly modify $P_{mn}(s)/Q_{mn}(s)$ to $\hat{P}_{mn}(s)/\hat{Q}_{mn}(s)$ such that $\hat{P}_{mn}(s)/\hat{Q}_{mn}(s)$ is C^1 continuous approximation of $Y(s)$ on $[-\beta, 0]$(see subsections (3.3.1) and (3.3.3) for details).

6. Denote the set of all the points $(\tilde{x}_i, \tilde{y}_i)$, $(\hat{x}_i, \hat{y}_i)$, the set T and the boundary points of $\mathbf{C}_B$ by V. The curve $\mathbf{C}_B$ yields a natural multigraph[1] G having V as its vertex set and the set of curve segments of $\mathbf{C}_B$ joining any pair of points in V, as its edge set E. Now starting from each (simple) point (x_i, y_i) in V we trace out the multigraph G, approximating each of its edges by C^1 continuous piecewise rational curves as in the following:
Compute the Taylor expansion that yields (without loss of generality) the single analytic branch given, by

$$\begin{aligned} X(s) &= x_i + s \\ Y(s) &= \sum_{j=0}^{\infty} c_j^{(i)} s^j, \quad c_0^{(i)} = y_i \end{aligned}$$

Exactly the same steps as above are used for determining the Padé approximation, β and modified Padé approximants for a C^1 ϵ-approximation of these analytic branches. The C^1 continuity here is achieved at the point $(\tilde{x}_i = X(\beta), \tilde{y}_i = Y(\beta))$ between the original curve and the ϵ-approximation and the multigraph is updated with only this single vertex till all edges are visited exactly once in the Euler tour. For each visited edge the C^1 piecewise approximation rational curves are stored in a separate list and finally output.

3 Details and Correctness of Algorithm

3.1 Expansion at Simple Points

Let $f(x, y) = \sum a_{ij} x^i y^j = 0$ be an algebraic curve and (x_0, y_0) be a simple point on it. By a simple translation $x = \tilde{x} + x_0, \;\; y = \tilde{y} + y_0$ we can move (x_0, y_0) to the origin, hence we assume that $(x_0, y_0) = (0, 0)$, i.e., $f(0, 0) = 0$. Since $(0, 0)$ is a simple point of the curve, we assume, without loss of generality, that $f_y(0, 0) = 1$. Let

$$f^0(x, y) = f(x, y) = y - a_1 x + a_0^0(x) + a_1^0(x) y + \ldots + a_n^0(x) y^n$$

[1] A graph with perhaps multiple edges between a pair of vertices

with $ord(a_0^0) > 1$. As a function of x, $y = y(x)$ has order ≥ 1. Let $y_1 = y - a_1 x$. Then the order of $y_1 = y_1(x)$ is ≥ 2. Let

$$f^1(x, y_1) = f^0(x, y_1 + a_1 x) = y_1 - a_2 x^2 + a_0^1(x) + a_1^1(x) y_1 + \ldots + a_n^1(x) y_1^n$$

then $ord(a_0^1) > 2$. Let $y_2 = y_1 - a_2 x^2$. Then we get $f^2(x, y_2) = f^1(x, y_2 + a_2 x^2)$. Repeat this procedure. We get $a_1 x, a_2 x^2, a_3 x^3 \ldots$. Then $\sum_{i=1}^{\infty} a_i x^i$ is the power series expansion.

This algorithm is very simple and easy to implement. If we want to compute $a_1 x, a_2 x^2 \ldots$ up to $a_k x^k$, then the terms in $a_i^j(x)$ with degree $> k-(j+1)^i$ can be killed during the computation, since these terms have no contribution to $\sum_{i=1}^{k} a_i x^i$. Hence the algorithm is very effective.

3.2 Expansion at Singular Points

We rely on the classical resolution of algebraic plane curves [1, 23] based on the Weierstrass Preparation Theorem [24] and Newton power series factorizations[17], using the technique of Hensel lifting[14]. We repeat it here for the sake of completeness.

3.2.1 Hensel Lifting

Consider $f(x, y)$ of degree d. Assume it is monic in y .

$$f(x,y) \;=\; f_0(y) \;+\; f_1(y)x \;+\; \cdots \;+\; f_k(y)x^k \;+\; \cdots$$

We wish to compute real power series factors $g(x, y)$ and $h(x, y)$ where $f(x, y) = g(x, y)h(x, y)$. The technique of Hensel lifting allows one to reconstruct the power series factors

$$\begin{aligned} g(x,y) &= g_0(y) + g_1(y)x + \cdots + g_i(y)x^i + \cdots \\ h(x,y) &= h_0(y) + h_1(y)x + \cdots + h_j(y)x^j + \cdots \end{aligned} \tag{3.1}$$

from initial factors $f(0, y) = f_0(y) = g_0(y)h_0(y)$.

Consider the factorization of $f(0, y) = f_0(y)$ as the base case of $k = 0$. Assume $f_0(y)$ is of degree d. Choose real coprime factors $g_0(y)$ of degree p and $h_0(y)$ of degree q satisfying: $p + q = d$. Real coprimeness is achieved by ensuring that g_0 and h_0 contain distinct real roots of f_0 and that complex conjugate pairs are not split up. However, it may arise that the only coprime factors of f_0 are complex, that is, the distinct roots are complex conjugates, in which case the curve $f(x, y) = 0$ does not intersect the y-axis and there is no real Newton power series factorization. Since $GCD(g_0(y), h_0(y)) = 1$ using the fast GCD algorithm we can also compute $\alpha(y)$ and $\beta(y)$ such that $\alpha(y)g_0(y) + \beta(y)h_0(y) = 1$.

In the iterative Case of $k \geq 1$, we compute $g_k(y)$ and $h_k(y)$ of the desired factorization (3.1), with degree of $g_k(y) < p$ and degree of $h_k(y) < q$, as follows. We note from (3.1) that

$$f_k(y) \;=\; \sum_{i\,+\,j\,=\,k} g_i(y)h_j(y)$$

and additionally

$$f_k(y) \;-\; \sum_{i>0,\; j>0,\; i+j=k} g_i(y)h_j(y) \;=\; g_0(y)h_k^*(y) \;+\; h_0(y)g_k^*(y) \tag{3.2}$$

Hence,

$$\begin{aligned} h_k^*(y) &= \alpha(y)[f_k(y) - \sum_{i>0,\ j>0,\ i+j=k} g_i(y)h_j(y)] \\ g_k^*(y) &= \beta(y)[f_k(y) - \sum_{i>0,\ j>0,\ i+j=k} g_i(y)h_j(y)] \end{aligned}$$

If degree $h_k^*(y) \geq q$ then compute $h_k(y) = h_k^*(y) \bmod h_0(y)$ and set $g_k(y) = \gamma(y)g_0(y) + g_k^*(y)$ where $h_k^*(y) = \gamma(y)h_0(y) + h_k(y)$.

$$f_k(y) - \sum_{i>0,\ j>0,\ i+j=k} g_i(y)h_j(y) = g_0(y)h_k(y) + h_0(y)g_k(y) \tag{3.3}$$

Clearly degree $h_k(y)$ is $< q$. Additionally in (3.3) the degree of $g_k(y)$ must also be $< p$. This is so because in (3.3) the degree of the LHS is $< d$ and since degree $g_0(y)h_k(y)$ is $< d$ and degree $h_0(y)$ is $= q$, it must be that degree $g_k(y)$ is $< p$.

3.2.2 Weierstrass Factorization

Consider $f(x,y)$ with degree d and $ord_y\ f(0,y) = e < \infty$. The $ord_y\ f(0,y)$ is the y-exponent of the lowest degree term in $f(0,y)$ and is equal to ∞ if $f(0,y) = 0$. The occurrence of $f(0,y) = 0$ can be rectified by a simple linear transformation (rotation) of $f(x,y)$, which avoids making the x-axis tangent to the curve $f(0,y) = 0$ at the origin, and hence yields a nonzero $f(0,y)$ and a finite $ord_y\ f(0,y)$. A Weierstrass power series factorization is of the form $f(x,y) = g(x,y)\underbrace{(y^e + a_{e-1}(x)y^{e-1} + \cdots + a_0(x))}_{h(x,y)}$ where $g(x,y)$ is a unit power series, i.e., $g(0,0) \neq 0$ while $h(x,y)$ is a "distinguished" polynomial in y with coefficients $a_i(x)$, $i = 0 \ldots e-1$ being non-unit power series, i.e., $a_i(0) = 0$. The Weierstrass preparation can be achieved via Hensel Lifting from the initial factors:

$$f(0,y) = f_0(y) = \underbrace{(a_0 + a_1 y + \cdots)}_{g_0(y)}\underbrace{y^e}_{h_0(y)}, \qquad a_0 \neq 0$$

3.2.3 Newton Factorization

Consider $h(x,y)$, a monic polynomial in y of degree e, with no repeated factors and with coefficients polynomial or power series or meromorphic series in x (like the "distinguished" polynomial of the Weierstrass factorization)

$$h(x,y) = y^e + a_{e-1}(x)y^{e-1} + \cdots + a_0(x)$$

Then it is possible to factor $h(x,y)$ into real linear factors of the type

$$h(x,y) = \Pi_{i=1}^{e}\ (y - \eta_i((t)))$$

with $t^m = x$ and m a positive integer and $\eta_i((t))$ a real power series or meromorphic series. This factorization is also achieved via Hensel lifting. We precondition the curve so that it admits a non-trivial base factorization, i.e. having at least two real coprime factors which can be lifted.

Step 1: Cancel the term $a_{e-1}(x)$ via the substitution $\tilde{y} = y + \frac{a_{e-1}(x)}{e}$. Note, that the case when all other $a_i(x)$ terms also vanish under this substitution is when the original $h(x, y) = (y - \frac{a_{e-1}(x)}{e})^e$ (a repeated factor which does not occur for our input curves).

Step 2: Ensure some $a_{e-i}(0) \neq 0$ for $i \geq 2$ via the substitution $\breve{y} = \frac{\tilde{y}}{x^\lambda}$ with $\lambda = min_{(2 \leq i \leq e)} \frac{\alpha_i}{i}$ and $\alpha_i = ord_x a_{e-i}(x)$. Then $h(0, \breve{y}) = h_0(\breve{y})$ has at least two distinct roots. If the only roots are complex, return "No real branches at the origin" and skip Step 3.

Step 3: Use Hensel lifting to lift the factorization $h_0(\breve{y}) = g_0(\breve{y})\, \breve{h}_0(\breve{y})$ with g_0 being linear, to $h(x, \breve{y}) = g(x, \breve{y})\, \breve{h}(x, \breve{y})$ and apply the inverse of the coordinate substitutions in Steps 1 and 2. Repeat Steps 1-3 until all factors of h are linear or all real factors are obtained.

3.3 C^1 Continuous Padé Approximation

3.3.1 C^1-continuity—Approach I

Let $P_{mn}(s)/Q_{mn}(s)$ be the (m, n) Padé approximation of $Y(s)$, That is $\frac{P_{mn}(s)}{Q_{mn}(s)} - Y(s) = O(s^{m+n+1})$. Let $\beta > 0$ be a real number, corresponding to a point on the analytic branch of the original curve C, such that $\frac{P_{mn}(s)}{Q_{mn}(s)}$ is analytic for $s \in [0, \beta]$. This β is determined in subsection 3.3.2.

Consider

$$\frac{\tilde{P}_{mn}(s)}{Q_{mn}(s)} = \frac{P_{mn}(s) + s^k(a + bs)}{Q_{mn}(s)}, \qquad 2 \leq k < m \tag{3.4}$$

Note that the above choice of $\tilde{P}_{mn}(s)$ change neither the degree of the approximation nor the order of the approximation error (shown in subsection 3.3.2). On the other hand, it is easy to see that

$$Y(s) - \frac{\tilde{P}_{mn}(s)}{Q_{mn}(s)} = O(s^k)$$

Any choice of k within the allowed range suffices and we have currently left that as a parameter in our implementation. For a fixed choice of k, determine a and b such that

$$\frac{\tilde{P}_{mn}(\beta)}{Q_{mn}(\beta)} = Y(\beta), \qquad C^0 \text{ continuity} \tag{3.5}$$

and further

$$\left(\frac{\tilde{P}_{mn}}{Q_{mn}}\right)'(\beta) = Y'(\beta), \qquad C^1 \text{ continuity} \tag{3.6}$$

Hence, for C^0 continuity, we have

$$a = \frac{Y(\beta)Q_{mn}(\beta) - P_{mn}(\beta)}{\beta^k} \tag{3.7}$$

and $b = 0$. For C^1 continuity, it follows from (3.5) and (3.6) that

$$a + b\beta = \frac{Y(\beta)Q_{mn}(\beta) - P_{mn}(\beta)}{\beta^k} \tag{3.8}$$

$$ka + (k+1)b\beta = \frac{(YQ_{mn} - P_{mn})'(\beta)}{\beta^{k-1}} \tag{3.9}$$

Since the matrix $\begin{bmatrix} 1 & \beta \\ k & (k+1)\beta \end{bmatrix}$ is nonsingular for $\beta \neq 0$, equations (3.8) and (3.9) have a unique solution and

$$a = \frac{(k+1)(YQ_{mn} - P_{mn})(\beta) - \beta(YQ_{mn} - P_{mn})'(\beta)}{\beta^k}$$

$$b = \frac{\beta(YQ_{mn} - P_{mn})'(\beta) - k(YQ_{mn} - P_{mn})(\beta)}{\beta^{k+1}}$$

For C^{-1} continuity , i.e no continuity constraints, $a = b = k = 0$. For C^0 continuity , i.e no gaps, $b = 0$ and a is computed as (3.7) for some fixed k such that $1 \leq k \leq m$.

3.3.2 Approximation Error Bound—Approach I

We now compute $\beta > 0$ a real number, corresponding to a point on the analytic branch of the original curve **C**, such that the segment $\frac{P_{mn}(s)}{Q_{mn}(s)}$ is analytic for $s \in [0, \beta]$. The following error analysis is based on the functional distance between the curve branch and the approximating segment. Similar error analysis can also be achieved for more geometric distance measures.

Note that $\tilde{P}_{mn}(s) = P_{mn}(s) + s^k(a + bs)$, where a and b are chosen to enforce C^1 continuity. Since

$$Y(s) - \frac{\tilde{P}_{mn}(s)}{Q_{mn}(s)} = \frac{Y(s)Q_{mn}(s) - P_{mn}(s) - s^k(a+bs)}{Q_{mn}(s)}$$

$s^k(a+bs)$ can be regarded as an C^1 interpolating polynomial of $Y(s)Q_{mn}(s) - P_{mn}(s)$ at points 0 and β. Hence we have

$$Y(s) - \frac{\tilde{P}_{mn}(s)}{Q_{mn}(s)} = \frac{(YQ_{mn} - P_{mn})^{(k+2)}(\xi)}{Q_{mn}(s)(k+2)!} s^k(s-\beta)^2, \quad \xi \in (0, \beta)$$

where $(YQ_{mn} - P_{mn})^{(k+2)}$ is the $(k+2)^{th}$ derivative of the power series. Since

$$|s^k(s-\beta)^2| \leq \frac{4k^k\beta^{k+2}}{(k+2)^{(k+2)}}, \qquad s \in [0, \beta]$$

we have

$$\left| Y(s) - \frac{\tilde{P}_{mn}(s)}{Q_{mn}(s)} \right| \leq \left| \frac{(YQ_{mn} - P_{mn})^{(k+2)}(\xi)}{Q_{mn}(s)(k+2)!} \right| \frac{4k^k\beta^{k+2}}{(k+2)^{(k+2)}}$$

From $(YQ_{mn} - P_{mn})(s) = \sum_{i=m+n+1}^{\infty} e_i s^i$, we have

$$\left| \frac{(YQ_{mn} - P_{mn})^{(k+2)}(\xi)}{(k+2)!} \right| \leq \sum_{i=m+n+1}^{\infty} |e_i| a_{k+2}^i \beta^{i-k-2}$$

Let $Q_{mn}^{-1}(s) = \sum_{i=0}^{\infty} q_i s^i$ and $|Q_{mn}^{-1}(s)| \le \sum_{i=0}^{\infty} |q_i|\beta^i$, then

$$\left|\frac{(YQ_{mn} - P_{mn})^{(k+2)}(\xi)}{Q_{mn}(s)(k+2)!}\right| \le \left|\frac{(YQ_{mn} - P_{mn})^{(k+2)}(\xi)}{(k+2)!}\right| |Q_{mn}^{-1}(s)| = (\sum_{i=0}^{\infty} r_i\beta^i)\beta^{m+n-k-1}$$

Therefore from the above analysis and the previous subsection we have
Theorem 1. *Let*

$$\sum_{i=0}^{\infty} r_i\beta^i = (\sum_{i=m+n+1}^{\infty} |e_i|a_{k+2}^i\beta^{i-m-n-1})(\sum_{i=0}^{\infty} |q_i|\beta^i)$$

Then

$$1^* \quad \left(\frac{\tilde{P}_{mn}}{Q_{mn}}\right)^{(i)}(0) = Y^{(i)}(0), \quad i = 0, 1, \cdots, k-1.$$

$$2^* \quad \left(\frac{\tilde{P}_{mn}}{Q_{mn}}\right)(\beta) = Y(\beta), \quad \left(\frac{\tilde{P}_{mn}}{Q_{mn}}\right)^{(1)}(\beta) = Y^{(1)}(\beta).$$

$$3^* \quad \left|Y(s) - \frac{\tilde{P}_{mn}(s)}{Q_{mn}(s)}\right| \le (\sum_{i=0}^{\infty} r_i\beta^i)\frac{4k^k\beta^{m+n+1}}{(k+2)^{(k+2)}}, \quad s \in [0, \beta]. \tag{3.10}$$

An interesting property of (3.10) is that the order of the approximation does not depends on k. Further, the error bound depends on P_{mn}/Q_{mn} but not a and b. Hence (3.10) can be used to compute the approximation range β after Padé approximation is obtained. For C^0 continuity, a similar bound can be obtained.

In our implementation, we take $\sum_{i=0}^{\infty} r_i\beta^i \approx r_0 + r_1\beta$ and then determine β_1 such that

$$(r_0 + r_1\beta_1)\frac{4k^k\beta_1^{m+n+1}}{(k+2)^{(k+2)}} \le \epsilon$$

Next compute the smallest pole of the rational function, i.e.

$$\beta_2 = \zeta * \min\{z_i : Q_{mn}(z_i) = 0, \quad z_i \in I\!R\}$$

for some positive constant $\zeta < 1$ and take $\beta_3 = \min\{\beta_1, \beta_2\}$. From the point on the Padé approximation $(X(\beta_3), P_{mn}(\beta_3)/Q_{mn}(\beta_3))$ we compute via Newton's method the nearest point $(\tilde{x}_i, \tilde{y}_i)$ on the analytic branch

$$\begin{aligned} X(s) &= x_i + s^{k_i} \\ Y(s) &= \textstyle\sum_{j=0}^{\infty} c_j^{(i)} s^j \end{aligned}$$

of the original curve $f(x, y) = 0$. Finally we determine β from the equation $\beta^{k_i} = \tilde{x}_i - x_i$.

3.3.3 C^1-continuity—Approach II

The second method for getting C^1 (or C^0) continuous Padé approximation is to first modify the $Y(s)$ as $\tilde{Y}(s)$:

$$\tilde{Y}(s) = \sum_{i=0}^{\infty} \tilde{c}_i s_i = \begin{cases} Y(s) & \text{for } C^{-1} \\ Y(s) + as^{m+n} & \text{for } C^0 \\ Y(s) + s^{m+n-1}(b + as) & \text{for } C^1 \end{cases} \tag{3.11}$$

and then compute (m, n) Padé approximation $P_{mn}(s)/Q_{mn}(s)$ for $\tilde{Y}(s)$. After that determine a and b such that $P_{mn}(s)/Q_{mn}(s)$ is C^1 (or C^0) continuous on $[0, \beta]$. If $n = 0$, the problem is reduced to the approach one of last subsection with $k = m - 1$. Now assume $n > 1$. From the expression of Padé approximation:

$$P_{mn}(s, a, b) = det \begin{bmatrix} c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m+n} + a & c_{m+n-1} + b & \cdots & c_m \\ \sum_{i=0}^{m} \tilde{c}_i s^i & \sum_{i=0}^{m-1} \tilde{c}_i s^{i+1} & \cdots & \sum_{i=0}^{m-n} \tilde{c}_i s^{i+n} \end{bmatrix}$$

$$Q_{mn}(s, a, b) = det \begin{bmatrix} c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m+n} + a & c_{m+n-1} + b & \cdots & c_m \\ 1 & s & \cdots & s^n \end{bmatrix}$$

we know that $P_{mn}(s, a, b)$, $Q_{mn}(s, a, b)$ is linear in a, degree at most 2 in b. Then by

$$\left(\frac{P_{mn}(s)}{Q_{mn}(s)}\right)_{s=\beta} = Y(\beta), \quad \left(\frac{P_{mn}(s)}{Q_{mn}(s)}\right)'_{s=\beta} = Y'(\beta)$$

We have

$$\begin{cases} P_{mn}(\beta, a, b) - Y(\beta)Q_{mn}(\beta, a, b) = 0 \\ P'_{mn}(\beta, a, b) - Y(\beta)Q'_{mn}(\beta, a, b) - Y'(\beta)Q_{mn}(\beta, a, b) = 0 \end{cases} \tag{3.12}$$

From the first equation, we get $a = d_0 b^2 + d_1 b + d_2$ for some constant $d's$. Substitute a into the second equation, we get degree 2 equation $a_0 b^2 + a_1 b + a_2 = 0$ of b. It has two solutions. At this moment one may immediately ask the question: Which solution do you choose from? Is the problem of modified Padé approximation equivalent to rational Hermite interpolation problem at points 0 and β? Now we try to answer the questions.

Lemma 1. *Denote*

$$A(m/n) = det \begin{bmatrix} c_m & c_{m-1} & \cdots & c_{m-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m+n-1} & c_{m+n-2} & \cdots & c_m \end{bmatrix}$$

Then if $A(m-1/n-1)$ is nonsingular when $n > 1$, then there exists uniquely a and b such that

$$rank \begin{bmatrix} c_{m+1} & c_m & \cdots & c_{m-n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m+n} + a & c_{m+n-1} + b & \cdots & c_m \end{bmatrix} < n, \quad for \;\; n > 0$$

Proof. If $n = 1$, take $a = -c_{m+n}, b = -c_{m+n-1}$. Then the matrix considered is reduced to zero. Hence the needed conclusion holds. Now suppose $n > 1$ and let A be the above matrix and A_1 be the last n columns of A. Then by the assumption of the lemma we know that there exists uniquely

$$b = (-1)^n \frac{A(m/n)}{A(m-1/n-1)}$$

such that the matrix A_1 is singular. For such b there exists uniquely a such that the last row of A can be expressed by the first $n-1$ rows and this complete the proof of the lemma. ◇

From the expression of Padé approximation and this lemma we know that there exists uniquely a and b such that $P_{mn} = Q_{mn} = 0$. Therefore equation (3.12) is satisfied by this trivial solution and this solution is not what we wanted. We should choose the solution which does not make Q_{mn} to be zero. After knowing which solution we should choose, we can answer the second question. Since the solution of the rational Hermite interpolation problem is unique, the modified Padé approximation must be the rational Hermite interpolant. On the other hand, expanding the rational Hermite interpolant into power series at origin and then computing the Padé approximant of the power series, we would get the same rational function by the uniqueness of the Padé approximation. Hence the the modified Padé approximation problem is equivalent to the rational Hermite interpolation problem. The computation approach here is easier than the known methods for computing the rational Hermite interpolant.

Theorem 2. *Let* $R_{mn}(s,a,b) = P_{mn}(s,a,b)/Q_{mn}(s,a,b)$ *be the Padé approximation of* $\tilde{Y}(s)$ $= \sum_{i=0}^{\infty} \tilde{c}_i s^i$ *defined as (3.11) and* $A(m-1/n-1)$ *and* $A(m/n)$ *are not zero.*

1. *Then for any* b*, exists* a *such that*

$$R_{mn}(\beta, a, b) = Y(\beta), \quad \beta \neq 0 \tag{3.13}$$

   if $Q_{m-1,n-1}(\beta) \neq 0$ *and* $r_{m-1,n-1}(\beta) \neq 0$.

2. *If the above condition is satisfied, then there exist* a *and* b *such that*

$$R'_{mn}(\beta, a, b) = Y'(\beta), \quad \beta \neq 0 \tag{3.14}$$

   if $\left(\frac{s r_{m-2,n-2}(s)}{r_{m-1,n-1}(s)} \right)'_{s=\beta} \neq 0$ *and*

$$A^2(m-1/n-1) \left(\frac{r_{m,n}(s)}{s r_{m-1,n-1}(s)} \right)'_{s=\beta} \neq A^2(m/n) \left(\frac{s r_{m-2,n-2}(s)}{r_{m-1,n-1}(s)} \right)'_{s=\beta} \tag{3.15}$$

   where $r_{mn}(s) = Y(s)Q_{mn}(s) - P_{mn}(s)$ *and* $A^2 = A * A$.

Proof. From the determinant expression of Padé approximation, we have

$$P_{mn}(s,a,b) = (-1)^{n-1} a s P_{m-1,n-1}(s) + b^2 s (s P_{m-2,n-2}(s)) + \cdots + P_{mn}(s)$$
$$Q_{mn}(s,a,b) = (-1)^{n-1} a s Q_{m-1,n-1}(s) + b^2 s (s Q_{m-2,n-2}(s)) + \cdots + Q_{mn}(s)$$

where $\cdots$ part is a linear monomial in b and $(sP_{m-2,n-2}(s), sQ_{m-2,n-2}(s)) = (-s^{m-1}, 0)$ for $n = 1$. Then (3.13) holds if $ar_{m-1,n-1}(\beta) = (-1)^n[b^2\beta r_{m-2,n-2}(\beta) + \cdots + r_{mn}(\beta)]$. Hence there exists a such that this equality holds for any b if $r_{m-1,n-1}(\beta) \neq 0$. This is conclusion one.

Substitute P'_{mn} and Q'_{mn} into the second equation of (3.12), after some computations, one get the following equation

$$(-1)^{n-1}a(sr_{m-1,n-1}(s))' + b^2(s^2 r_{m-2,n-2}(s))' + \cdots + (r_{mn}(s))' = 0$$

It follows from

$$a = (-1)^n \left[\frac{\beta^2 r_{m-2,n-2}(\beta)}{\beta r_{m-1,n-1}(\beta)} b^2 + \cdots + \frac{r_{mn}(\beta)}{\beta r_{m-1,n-1}(\beta)}\right]$$

that

$$\left(\frac{s^2 r_{m-2,n-2}(s)}{s r_{m-1,n-1}(s)}\right)'_{s=\beta} b^2 + \cdots + \left(\frac{r_{mn}(s)}{s r_{m-1,n-1}(s)}\right)'_{s=\beta} = 0 \tag{3.16}$$

From Lemma 1 we know that this equation has a trivial solution which makes $P_{mn} = Q_{mn} = 0$. It has two solutions if and only if the highest coefficient of the equation is not zero. Let the quadratic equation (3.16) be denoted as $\alpha_2 b^2 + \alpha_1 b + \alpha_0 = 0$ with roots r_1 and r_2 then since $r_1 * r_2 = \frac{\alpha_0}{\alpha_2}$, we know that in order to make the two solutions of (3.16) be distinct, it suffices to satisfy the inequality (3.15). $\diamond$

4 The Computation of the Singularity

The computation of the singularity consists of two sub-problems. One is to find the singular points, the other is to determine the order of the singular points. The singular points computed should have good accuracy such that the correct order can be determined from which.

For finding the singular points, we solve the equations

$$\begin{cases} f(x, y) = 0 \\ \alpha f_x(x, y) + \beta f_y(x, y) = 0 \end{cases} \tag{4.1}$$

using multivariate resultants and based on the method of birational maps [5], where the constants α and β are chosen such that f and $\alpha f_x + \beta f_y$ are coprime. In this method, we are led to solving a system of equations in the following form:

$$\begin{cases} \phi_0(X) = 0 \\ Y = \phi_1(X) \end{cases} \tag{4.2}$$

with (X, Y) and (x, y) are linearly related, where ϕ_0 is a polynomial and ϕ_1 is a rational function. The first equation of (4.2) can be solved by calling C-library to get the initial approximate values and then using iterative methods to get the higher precision solutions. Then the solution of (4.1) is received by the second equation of (4.2) and the linear relation between (X, Y) and (x, y).

In this approach, the equation (4.2) is produced by symbolic computation. After the initial value is got by numerical methods, the refinement afterwards is also symbolic. In the development of the following, we shall determine the required precision of the refinement. This precision will guarantee that the order of the singular point is correctly determined.

Let $p^* = (x^*, y^*)$ be a singular point of $f(x, y) = 0$. Then the order of it is the minimal integer k, for which $f_{ij}(p^*) \neq 0$ for at least one pair (i, j) with $i + j = k$, where $f_{ij} = \frac{\partial^{i+j} f}{\partial x^i \partial x^j}$. In order to determine the correct order from the approximate singular point of p^*, we need to know the minimal value of $|f_{ij}(p^*)|$ if it is nonzero. The lower bound of this value can be estimated by the following gap theorem:

Gap Theorem[9]. *Let $\mathcal{P}(d, c)$ be the class of integral polynomials of degree d and maximum coefficient magnitude c. Let $f_i(x_1, \cdots, x_n) \in \mathcal{P}(d, c)$, $i = 1, \cdots, n$ be a collection of n polynomials in n variables which has only finitely many solutions when homogenized. If $(\alpha_1, \cdots, \alpha_n)$ is a solution of the system, then for any j either $\alpha_j = 0$, or $|\alpha_j| > (3dc)^{-nd^n}$*

For a given integer pair (i, j), let $z^* = f_{ij}(p^*)$, then using Gap Theorem to the following system:

$$\begin{cases} f(x, y) = 0 \\ f_x(x, y) = 0 \\ z - f_{ij}(x, y) = 0 \end{cases} \tag{4.3}$$

with a known solution (x^*, y^*, z^*), we know that if $z^* \neq 0$, then $|z^*| > (3dc_{ij})^{-3d^3}$, where d is the degree of f and c_{ij} is the maximum coefficient magnitude of the left hand side of (4.3). By this inequality, we can get the following criterion for testing whether $f_{ij}(p^*)$ is zero:

Test Criterion. *Let $gap = (3dc_{ij})^{-3d^3}$*

$$|f_{ij}(p) - f_{ij}(p^*)| < \frac{1}{2} gap \tag{4.4}$$

then $f_{ij}(p^*) = 0$ if and only if $|f_{ij}(p)| < \frac{1}{2} gap$.

Proof. If $|f_{ij}(p)| < \frac{1}{2} gap$, then

$$|f_{ij}(p^*)| \leq |f_{ij}(p)| + |f_{ij}(p) - f_{ij}(p^*)| < gap$$

Hence $f_{ij}(p^*) = 0$. On the other hand, if $f_{ij}(p^*) = 0$, then (4.4) implies the required inequality.

The Precision of the Singular Points. It follows from the test criterion above that, for knowing whether $f_{ij}(p^*) = 0$ or not, the computed singular point should have such a precision that the inequality (4.4) holds. Suppose the singular point is in the given bounding box $B = \{(x, y) : \; a_1 \leq x \leq a_2, b_1 \leq y \leq b_2\}$ in which the rational approximations are constructed, then by the Mean Value theorem, we have

$$|f_{ij}(p) - f_{ij}(p^*)| \leq \sqrt{f_{i+1,j}(p + \theta p^*) + f_{i,j+1}(p + \theta p^*)} \|p - p^*\| \leq M_{ij} \|p - p^*\|$$

where $\theta \in (0, 1)$, $M_{ij} = \sqrt{\sum_{ij} (|b_{i+1,j}| s + |b_{i,j+1}| t) s^i t^j}$, b_{ij} is defined by $f_{ij}(x, y) = \sum_{ij} b_{ij} x^i y^j$, $s = max\{|a_1|, |a_2|\}$ and $t = max\{|b_1|, |b_2|\}$. Therefore, (4.4) holds if

$$\|p - p^*\| \leq \frac{gap}{2M_{ij}}$$

That is, the computation of p^* should make p^* to have the accuracy $\frac{gap}{2M_{ij}}$ and therefore the computation should use $-log(\frac{gap}{2M_{ij}})/log2$ binary bits.

5 Implementation Issues & Examples

The rational approximation algorithms has been implemented in its entirety as part of GANITH, an X-11 based interactive algebraic geometry toolkit, using Common Lisp for the symbolic computation and C for all numeric and graphical computation. The input curve is assumed to have integral coefficients, which are considered to be exact. Floating point coefficients are allowed in the input curve representations, which are then converted to rational numbers and then converted to integers.

The Hensel power series computations of section 3, as well as its use in sections Weierstrass and Newton factorizations are based on a robust implementation of the fast euclidean HGCD algorithm [8]. Rational Padé approximants are also computed based on the same HGCD algorithm, [8]. Power Series are stored as truncated sparse polynomials, as are the polynomials representing the original algebraic curves, in recursive canonical form. In this form, a polynomial in the variables $x_1, \ldots, x_n$ is represented either as a constant, or as a polynomial in x_n whose coefficients are (recursively) polynomials in the remaining variables $x_1, \ldots, x_{n-1}$. A strength of this form (for purposes of implementation) is that multivariates "look like" univariates, making it easy to modify algorithms for univariate polynomials to handle multivariates. All these computations can be numerical.

In Newton factorizations, user options are provided to compute only real branch factorizations. This is achieved by not allowing complex conjugate roots of the appropriate univariate polynomial, to split in the base case of the Henselian computation. Singularity computations(see section 4) as well as the extreme points computations are done in GANITH using multivariate resultants and based on the method of birational maps [5]. The intersection points of the curve with the bounding box are computed by letting x or y to be constant and then solving one unknown equation. In these computations, symbolic as well as numerical computations are used.

Examples from the software implementation, are shown in Figure 5.1. The figure shows C^1 continuous piecewise rational quadratic and cubic approximations of several degree four and degree six algebraic plane curves. The upper left picture is a C^1 approximation of the plane curve $y^4 - 2y^3 + y^2 - 3x^2y + 2x^4 = 0$ by $(2,1)$-rational parametric curve segments with $\epsilon = 0.1$. The upper right picture is a C^1 approximation of the plane curve $y^2 - xy^2 - 2x^2y + x^2y^2 + x^4 = 0$ by $(3,3)$-rational parametric curve segments with $\epsilon = 0.1$. The lower left picture is a C^1 approximation of the plane curve $(x^2+y^2)^2 + 3x^2y - y^3 = 0$ by $(2,1)$-rational parametric curve segments with $\epsilon = 0.09$. The lower right picture is a C^1 approximation of the plane curve $(x^2+y^2)^3 - 4x^2y^2 = 0$ by $(2,1)$-rational parametric curve segments with $\epsilon = 0.1$.

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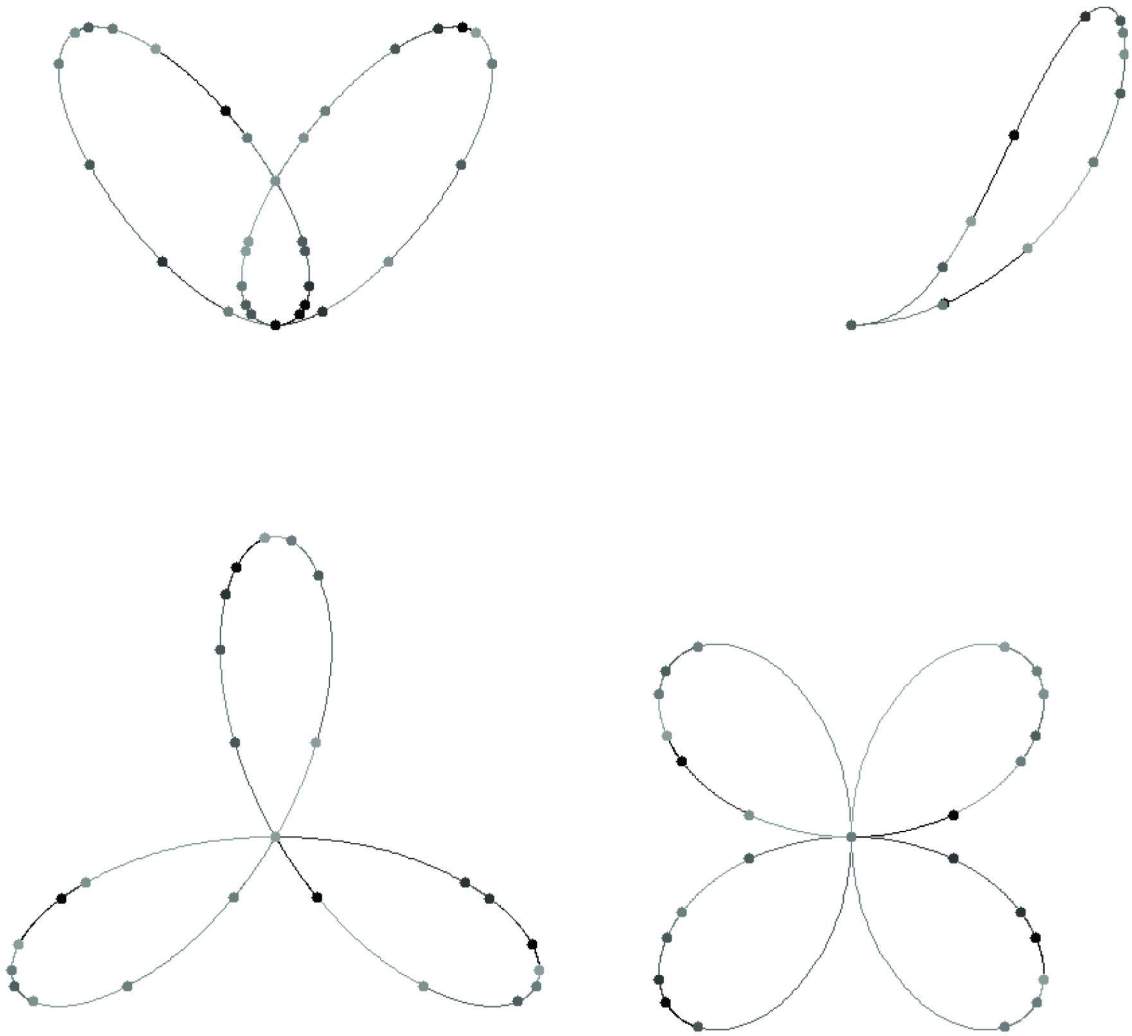

Figure 5.1: Piecewise Rational Approximations of Real Algebraic Curves

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