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Regular algebraic curve segments (I)—Definitions and characteristics

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Abstract

In this paper (part one of a trilogy), we introduce the concept of a discriminating family of regular algebraic curves (real, nonsingular and connected). Several discriminating families are obtained which yield different characterizations of the Bernstein–Bézier (BB) bivariate polynomials over the plane triangle and the quadrilateral domain such that their zero contours are smooth and connected. These regular curve segments in BB basis can be smoothly joined together to form algebraic curve splines or A-splines. Algorithms for the efficient graphics display of these new A-spline families are also provided.

Keywords: Algebraic curve; Discriminating family; D-regular; BB-form

1. Introduction

Consider an algebraic curve defined by a real polynomial equation in Bernstein–Bézier (BB) form over a plane triangle or a quadrilateral. Notwithstanding the overwhelming popularity of parametric curves in computer aided geometric design (CAGD), one is increasingly aware that curves defined by implicit equations are useful for certain CAGD operations (Bajaj, 1993). The primary drawback for the widespread use of the implicit algebraic curves is that the real curve may have singularities (e.g., cuspidal cubic), and may be disconnected (e.g., hyperbola) in a given region of the plane. For example, if we fit

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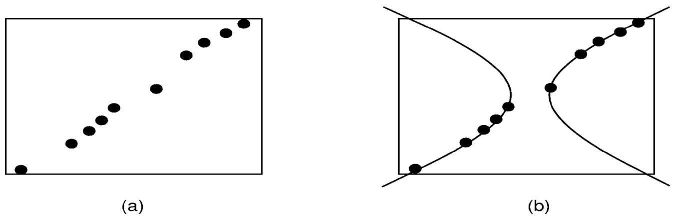

Fig. 1. (a) The points to be fitted; (b) The fitting curve.

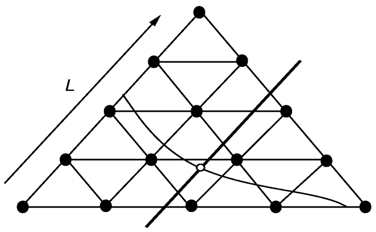

Fig. 2. The coefficients on the lines, that parallel to L, increase.

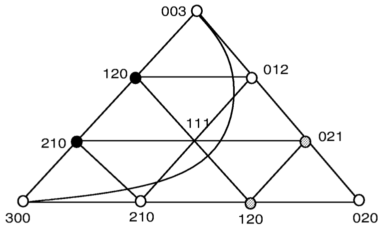

Fig. 3. Cubic: The real dots are positive. Shaded are negative. Empty are zero.

a cluster of points as shown in Fig. 1(a) with a quadratic, we often lead to the result shown in Fig. 1(b), if no additional conditions are imposed on the curve.

Hence, for some applications in CAGD, such as data fitting or shape design, it is natural to require that the curve be connected in a specified region.

In this paper (part I of III), we focus on isolating a regular piece of an algebraic curve that is defined on a given triangle or a given quadrilateral in BB-form. We introduce the concept of a discriminating family of regular algebraic curve segments (real, nonsingular and connected). Several discriminating families are obtained which yield different characterizations of the Bernstein–Bézier (BB)-form bivariate polynomials over the plane triangle and the quadrilateral domain such that their zero contours are smooth and single-sheeted. Furthermore, using these discriminating families, we can efficiently evaluate the algebraic curve segments for display. In parts II and III of this trilogy of papers, we consider the problems of interpolation and approximation by splines of regular algebraic curve segments (Xu et al., 2000) and their applications in scattered and dense data fitting (Bajaj and Xu, 2000).

As mentioned earlier, the main difficulties in dealing with real algebraic curves are the problems of real singularities and discontinuities. In attempts to overcome these difficulties, Sederberg in (Sederberg et al., 1985) set conditions on the coefficients of the BB-form of an implicitly defined bivariate polynomial on a triangle in such a way that if the coefficients on the lines that are parallel to one side, say L, of the triangle all increase (or decrease) monotonically in the same direction, then any line parallel to L will intersect the algebraic curve segment at most once (see Fig. 2). In (Sederberg et al., 1988), Sederberg, Zhao and Zundel give another similar set of conditions which guarantees the single-sheeted property of their TPAC by requiring that $\beta_{i0} \geqslant 0$, that $\beta_{0i}, \beta_{m-1,i} \leqslant 0$, and that the directional derivative of PAC (piecewise algebraic curves) with respect to any direction $s = \alpha u$ be non-zero within the triangle domain, where β_{ij} denotes the Bézier coefficients. Papers of Paluszny and Patterson (1992, 1993) construct G^1 and G^2 continuous cubic algebraic splines by using the cubic $F(\alpha_1, \alpha_2, \alpha_3) = \beta_{201}\alpha_1^2\alpha_3 + \beta_{102}\alpha_1\alpha_3^2 - \beta_{120}\alpha_1\alpha_2^2 - \beta_{021}\alpha_2^2\alpha_3 + \beta_{111}\alpha_1\alpha_2\alpha_3$ with $\beta_{201} > 0$, $\beta_{102} > 0$, $\beta_{120} > 0$, $\beta_{021} > 0$, and $(\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$ being barycentric coordinates (see Fig. 3). All the above characterizations dealing with BB triangles can be regarded as special cases of (Bajaj and Xu, 1999) in which the coefficients of the BB-form have a one-time sign change. The present paper is an extension of our paper (Bajaj and Xu, 1999) and deals with the BB-form on both the triangle and the quadrilateral. For the BB-form on the quadrilateral, a characterization for the single-sheeted purpose is given in (Patrikalakis and Kriezis, 1989) and is similar to Sederberg's in (Sederberg et al., 1988). In particular, if the coefficients increase or decrease monotonically in the x or y direction, then any line that is parallel to the x or y axis will intersect the curve at most once. This is again a special case of our characterization in this paper.

2. Notation and mathematical preliminary

Let $p_i = (x_i, y_i)^{\mathrm{T}} \in \mathbb{R}^2$ for $i = 1, \ldots, k$. Then $[p_1 p_2 \ldots p_k]$ denotes the *closed convex hull* of $\{p_i\}_{i=1}^k$. That is, $[p_1 p_2 \ldots p_k] = \{p \in \mathbb{R}^2\colon\ p = \sum_{i=1}^k \alpha_i p_i,\ 0 \leqslant \alpha_i \leqslant 1,\ \sum_{i=1}^k \alpha_i = 1\}$. If $0 < \alpha_i < 1$, then the set defined is the *open convex hull* of $\{p_i\}_{i=1}^k$, denoted by $(p_1 p_2 \ldots p_k)$. If $\alpha_i \in (-\infty, \infty)$, then the set defined is the *affine hull* of $\{p_i\}_{i=1}^k$, denoted by $\langle p_1 p_2, \ldots, p_k\rangle$. If $k = 3$ and p_1, p_2 and p_3 are affine independent, then $[p_1 p_2 p_3]$ is a triangle and $\alpha = (\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$ are known as barycentric coordinates which relate with the Cartesian coordinates $(x, y)^{\mathrm{T}}$ by

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{1}{\Delta(p_1, p_2, p_3)} \begin{bmatrix} y_2 - y_3, & x_3 - x_2 \\ y_3 - y_1, & x_1 - x_3 \end{bmatrix} \begin{bmatrix} x - x_3 \\ y - y_3 \end{bmatrix}$$

and $\alpha_3 = 1 - \alpha_1 - \alpha_2$ with

$$\Delta(p_1, p_2, p_3) = \det \begin{bmatrix} p_1 & p_2 & p_3 \\ 1 & 1 & 1 \end{bmatrix}.$$

On a triangle, the algebraic curve will be defined by the zero contour of

$$F_n(\alpha) = \sum_{i+j+k=n} \beta_{ijk} B_{ijk}^n(\alpha) \quad \text{with } B_{ijk}^n(\alpha) = C_{ij}^n \alpha_1^i \alpha_2^j \alpha_3^k, \tag{2.1}$$

where

$$C_{i_1 \cdots i_k}^n = \frac{n!}{i_1! \cdots i_k!(n - i_1 - \cdots - i_k)!}$$

(see Fig. 4 for cubic). If $k = 4$ and any three of p_i $(i = 1, \ldots, 4)$ are affine independent, then $[p_1 p_2 p_3 p_4]$ is a quadrilateral. We shall assume p_1, p_2, p_4 and p_3 are clockwise, and map $[p_1 p_2 p_3 p_4]$ to the unit square $S = [0, 1] \times [0, 1]$ in the uv-plane by

$$p = (p_1 + p_4 - p_2 - p_3)uv + (p_3 - p_1)u + (p_2 - p_1)v + p_1. \tag{2.2}$$

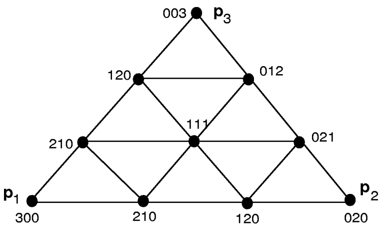

Fig. 4. Cubic coefficient index on the triangle $[p_1 p_2 p_3]$.

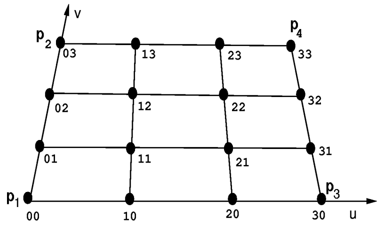

Fig. 5. Bi-cubic coefficient index on the quadrilateral $[p_1 p_2 p_3 p_4]$.

If $p_1 + p_4 = p_2 + p_3$, i.e., $[p_1 p_2 p_3 p_4]$ is a parallelogram, then (2.2) is linear. On a quadrilateral, the algebraic curve is defined by the zero contour of

$$G_{mn}(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \beta_{ij} B_i^m(u) B_j^n(v) \tag{2.3}$$

with $B_i^n(s) = C_i^n s^i (1-s)^{n-i}$ (see Fig. 5 for bi-cubic).

We could imagine that the discussion of the problem of isolating a single piece from the zero contour of $F_n(\alpha)$ or $G_{mn}(u, v)$ should relate to the problem of root isolation of a one-variable polynomial. Indeed, in our development, we are always led to the problem of judging a univariable polynomial equation having zero roots or one root in a given interval. The following set of lemmas is designed to solve this decision problem for different cases. For polynomials of low degree ($\leqslant 3$), we could give necessary and sufficient conditions. For other cases, we give sufficient conditions only.

Lemma 2.1. *Let $F(x) = \sum_{i=0}^n \beta_i B_i^n(x)$. If there exist an integer k $(0 < k < n)$ such that $\beta_i \geqslant 0$ for $i = 0, \ldots, k-1$ and $\beta_i \leqslant 0$ for $i = k+1, \ldots, n$, and there is at least one strict inequality in each set of the inequalities, then for any real $\alpha > 0$, the function $F_\alpha(x) = \sum_{i=0}^n \beta_i [B_i^n(x)]^\alpha$ has only one zero in the interval $(0, 1)$.*

Proof. If $\alpha = 1$, the lemma follows from the *variation diminishing property* (see p. 54 in (Farin, 1990)). Since $F_\alpha(x) = (1-x)^{n\alpha} \sum_{i=0}^n c_i t^i$ with $c_i = (C_i^n)^\alpha \beta_i$ and $t = (\frac{x}{1-x})^\alpha \in (0, \infty)$, then under the assumption of the lemma, the sequence $c_0, c_1, \ldots, c_n$ has a one-time sign change. It follows from Descarte's sign rule that the function $\sum_{i=0}^n c_i t^i$ has only one zero in $(0, \infty)$. Therefore, $F_\alpha(x)$ has only one zero in $(0, 1)$ (see Fig. 6). $\square$

Lemma 2.2. *$\sum_{i=0}^n \beta_i [B_i^n(x)]^\alpha \geqslant 0$ (or > 0) on $[0, 1]$, if and only if $\sum_{i=0}^n \beta_i$ $(C_i^n)^{\alpha-1} \times B_i^n(x) \geqslant 0$ (or > 0) on $[0, 1]$, where $\alpha > 0$.*

Proof. Let $t = \frac{x^\alpha}{x^\alpha + (1-x)^\alpha}$. Then for $x \in [0, 1]$ we have $t \in [0, 1]$ and

$$\sum_{i=0}^n \beta_i \left[B_i^n(x)\right]^\alpha = \left[x^\alpha + (1-x)^\alpha\right]^n \sum_{i=0}^n \frac{\beta_i (C_i^n)^\alpha x^{i\alpha} (1-x)^{(n-i)\alpha}}{[x^\alpha + (1-x)^\alpha]^n}$$

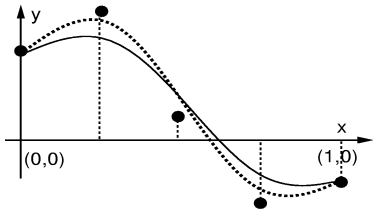

Fig. 6. $F(x)$(real line curve) and $F_\alpha(x)$(dotted line curve) for $n=4$.

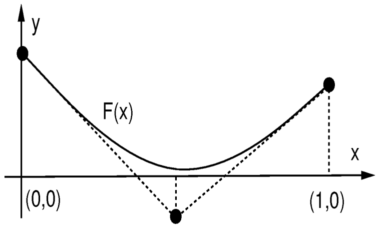

Fig. 7. Positive quadratic curve with a negative coefficient.

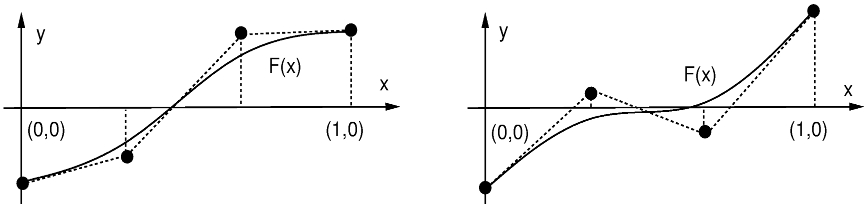

Fig. 8. Lemma 2.4: Necessary and sufficient conditions for a cubic to have one zero in $(0,1)$.

$$= \left[x^\alpha + (1-x)^\alpha\right]^n \sum_{i=0}^{n} \beta_i (C_i^n)^{\alpha-1} B_i^n(t).$$

The lemma follows directly from this equality. $\square$

Lemma 2.3. *Let* $F(x) = \sum_{i=0}^{2} \beta_i B_i^2(x)$. *Then* $F(x) > 0$ *on* $(0,1)$ *if and only if*

(i) $\beta_j \geqslant 0$ $(j = 0,1,2)$ *and at least one of* β_j *is positive, or*

(ii) $\beta_0 > 0,\ \beta_1 < 0,\ \beta_2 > 0$ *and* $\beta_1^2 < \beta_0\beta_2$.

Proof. It is obvious that $F(x) > 0$ on $(0,1)$ if condition (i) holds. Now suppose condition (ii) is satisfied. Then from $F'(r) = 0$ we get $r = (\beta_0 - \beta_1)/(\beta_0 + \beta_2 - 2\beta_1)$. Since $F''(r) = 2(\beta_0 + \beta_2 - 2\beta_1) > 0$, r is a minimum point of $F(x)$ in $(0,1)$ with the minimum value $(\beta_0\beta_2 - \beta_1^2)/(\beta_0 + \beta_2 - 2\beta_1)$. Hence $F(x) > 0$ on $[0,1]$ (see Fig. 7). This completes the proof of the sufficient part of the lemma. The necessary part can be proved similarly. $\square$

Lemma 2.4. *Let* $F(x) = \sum_{i=0}^{3} \beta_i B_i^3(x)$ *with* $\beta_0 < 0,\ \beta_3 > 0$. *Then* $F(x)$ *has a single zero in* $(0,1)$ *if and only if*

(i) $\beta_1 \leqslant 0$ *or* $\beta_2 \geqslant 0$, *or*

(ii) $\beta_1 > 0, \beta_2 < 0$ *and*

$$\Delta := \beta_0^2\beta_3^2 - 3\beta_1^2\beta_2^2 - 6\beta_0\beta_1\beta_2\beta_3 + 4\beta_0\beta_2^3 + 4\beta_1^3\beta_3 > 0.$$

Proof. (See Fig. 8.) If $\beta_1 \leqslant 0$ or $\beta_2 \geqslant 0$, then the sequence $\{\beta_i\}_{i=0}^3$ has one-time sign change. It follows from the *variation diminishing property* (see p. 54 in (Farin, 1990)) that

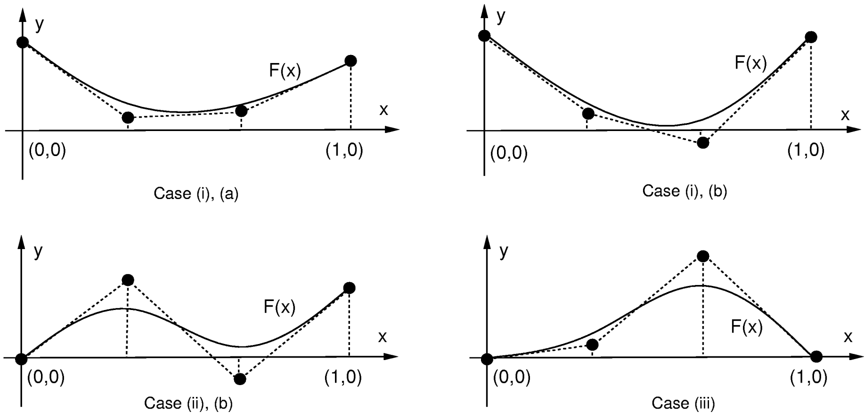

Fig. 9. Lemma 2.5: Necessary and sufficient conditions for a cubic to be positive.

$F(x)$ has a single zero in $(0,1)$. Suppose $\beta_1 > 0$ or $\beta_2 < 0$. Then $F(x)$ has no zero outside $[0,1]$. Let $F(x) = (1-x)^3 G(t)$, with $G(t) = \beta_0 + 3\beta_1 t + 3\beta_2 t^2 + \beta_3 t^3$ and $t = x/(1-x)$. Then since $G(-1) = (\beta_0 - \beta_3) + 3(\beta_2 - \beta_1) \neq 0$, $F(x)$ has one zero in $[0,1]$ if and only if $G(t)$ has one zero in $(-\infty, \infty)$. It is well known that $G(t)$ has one zero in $(-\infty, \infty)$ if and only if its discriminant $\Delta/(4\beta_3^4) > 0$. $\square$

Lemma 2.5. *Let* $F(x) = \sum_{i=0}^{3} \beta_i B_i^3(x)$. *Then*

(i) $F(x) > 0$ *on* $[0,1]$ *if and only if* (a) $\beta_0 > 0$, $\beta_3 > 0$, $\beta_1 \geqslant 0$ *and* $\beta_2 \geqslant 0$ *or* (b) $\beta_0 > 0$, $\beta_3 > 0$ *and* $\Delta > 0$.

(ii) $F(x) > 0$ *on* $(0,1]$ (*or* $[0,1)$) *and* $F(0) = 0$ (*or* $F(1) = 0$) *if and only if* (a) $\beta_0 = 0$, $\beta_3 > 0$ (*or* $\beta_3 = 0$ $\beta_0 > 0$), $\beta_1 \geqslant 0$ *and* $\beta_2 \geqslant 0$ *or* (b) $\beta_0 = 0$, $\beta_3 > 0$ (*or* $\beta_3 = 0$ $\beta_0 > 0$) *and* $\Delta > 0$.

(iii) $F(x) > 0$ *on* $(0,1)$ *and* $F(0) = F(1) = 0$ *if and only if* $\beta_0 = \beta_3 = 0$, $\beta_1 \geqslant 0$, $\beta_2 \geqslant 0$ *and* $\beta_1 + \beta_2 > 0$.

The proof of this lemma is similar to that of Lemma 2.4, and so we omit it here.

3. Discriminating families

Consider first the classical one variable C^1 function (it is of course defining a smooth curve) $y = g(x)$, $x \in [a,b]$. The smoothness of the curve $f(x,y) := y - g(x) = 0$ can be tested by considering if every straight line $x = \alpha$, $\alpha \in [a,b]$, intersects the curve only once (see Fig. 10(a)). The cases shown in Figs. 10(b)–(c) could not happen.

The essential point behind this observation is that if each line in the set $\{x = \alpha\colon \alpha \in [a,b]\}$ intersects the curve only once, then the curve is regular. That is, the family of these lines can be used to judge the regularity of a curve. Furthermore, in paper (Bajaj and Xu, 1999), we have used straight lines

$$\left\{\alpha(t) = (1-t)(\beta, 1-\beta, 0)^{\mathrm{T}} + t(0,0,1)^{\mathrm{T}},\ t \in [0,1]\colon\ \beta \in (0,1)\right\}$$

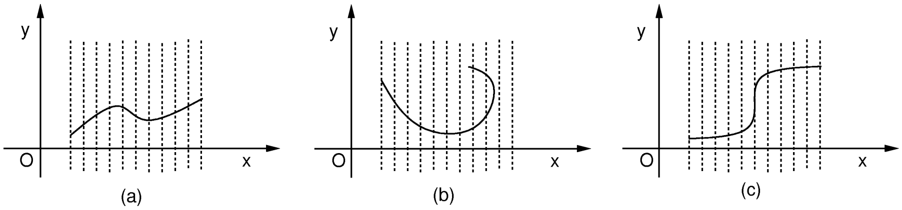

Fig. 10. (a) Single-valued regular function; (b) Multi-valued non-regular function; (c) Function not differentiable.

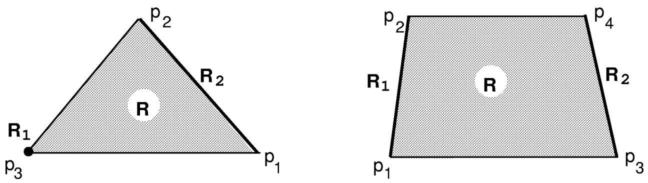

Fig. 11. Closed pieces R_1 and R_2 of the boundaries of a triangle and a quadrilateral.

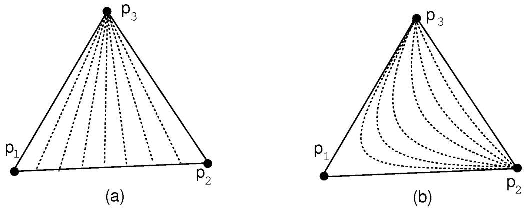

Fig. 12. (a) The lines D_1; (b) The quadratic family D_2.

in a triangle (see Fig. 12(a)) to intersect the curve. The conclusion we obtained there was that under certain conditions on the coefficients of a bivariate polynomial $F_n(\alpha)$, each line in this family will intersect the curve $F_n = 0$ only once in the triangle, and so the curve is regular. Again, this set of lines is used to judge the regularity of the curve. In this section, we extend these observations to introduce a general concept of a discriminating family. The purpose of the extension is that we want to find more algebraic curves that can be distinguished as regular.

The questions we raise are:

(1) Can we find simple discriminating families?

(2) Do simple conditions exist for the algebraic curve under which the curve can be judged as regular by the discriminating families?

To answer the first question, we give some examples of discriminating families after Definition 3.1. These families are then used to judge the regularities of various curves in the subsequent section. Hence, the answers to both the problems are affirmative.

Definition 3.1. For a given triangle or quadrilateral R, let R_1 and R_2 be two closed pieces of boundary of R with $R_1 \cap R_2 = \emptyset$ (see Fig. 11). Let $D = \{A_s(x, y) = \gamma(x, y) -$

$s\delta(x, y) = 0$: $s \in [0, 1]\}$ be an algebraic curve family with s as a parameter and $\delta(x, y) > 0$ on $R \setminus \{R_1, R_2\}$ such that

(1) Each curve in D passes through R_1 and R_2;
(2) Each curve in D is regular in the interior of R;
(3) For $\forall p \in R \setminus \{R_1, R_2\}$, there exists one and only one $s \in [0, 1]$ such that $A_s(p) = 0$.

Then we say D is a discriminating family on R, denoted by $D(R, R_1, R_2)$.

In the following, we shall give four examples of discriminating families. The first two are defined on the triangle, the other two are defined on the quadrilateral.

Example 3.1. Let

$$D_1\big([p_1p_2p_3], p_3, [p_1p_2]\big) = \big\{\alpha_2 - s(\alpha_1 + \alpha_2) = 0:\ s \in [0, 1]\big\},$$

where $(\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$ is the barycentric coordinate with respect to the triangle $[p_1p_2p_3]$. It is easy to see that $D_1([p_1p_2p_3], p_3, [p_1p_2])$ consists of straight lines (see Fig. 12(a)) in the triangle $[p_1p_2p_3]$ that connect the point p_3 and all the points on $[p_1p_2]$. Hence $D_1([p_1p_2p_3], p_3, [p_1p_2])$ is obviously a discriminating family.

Example 3.2. Let

$$D_2\big([p_1p_2p_3], p_2, p_3\big) = \big\{(1 - s)\alpha_2\alpha_3 - s\alpha_1^2 = 0:\ s \in [0, 1]\big\}.$$

$D_2([p_1p_2p_3], p_2, p_3)$ consists of quadratics (see Fig. 12(b)) and it is a discriminating family. The proof of this conclusion follows from Theorem 4.1.

Example 3.3. Let

$$D_3\big([p_1p_2p_3p_4], [p_1p_2], [p_3p_4]\big) = \big\{v = s:\ s \in [0, 1]\big\},$$

then $D_3([p_1p_2p_3p_4], [p_1p_2], [p_3p_4])$ is a discriminating family (see Fig. 13(a)).

Example 3.4. Let

$$D_4\big([p_1p_2p_3p_4], p_1, p_4\big) = \big\{(1 - s)u(1 - v) - s(1 - u)v = 0:\ s \in [0, 1]\big\}.$$

Then $D_4([p_1p_2p_3p_4], p_1, p_4)$ consists of hyperbola (see Fig. 13(b)) and it is a discriminating family. The proof of this fact follows from Theorem 4.5.

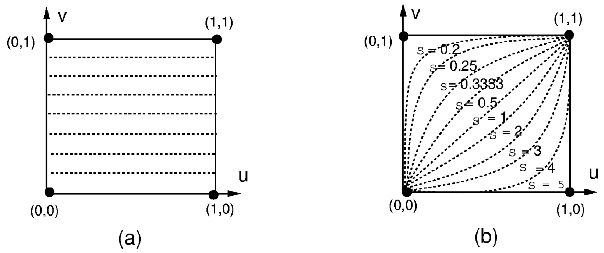

Fig. 13. (a) The lines D_3; (b) The hyperbolic family D_4.

Definition 3.2. For a given discriminating family $D(R, R_1, R_2)$, let $f(x, y)$ be a bivariate polynomial (or a C^1 continuous function on $R \setminus \{R_1, R_2\}$). If the curve $f(x, y) = 0$ intersects with each curve in $D(R, R_1, R_2)$ exactly once in the interior of R, we say the curve $f = 0$ is regular with respect to $D(R, R_1, R_2)$ (concisely stated as being $D(R, R_1, R_2)$-regular)).

Here we need to make the meaning of *intersect once* precise. Let $p^* = (x^*, y^*)^{\mathrm{T}}$ be a point on the curve $A_s(x, y) = 0$. Since the curve is regular, we represent it locally as $y = h(x)$ (or $x = g(y)$) in the neighborhood of p^*. The term *intersect once* means x^* is a single zero of $f(x, h(x))$.

It is easy to show that a $D(R, R_1, R_2)$-regular curve $f = 0$ is regular. In fact, if $\nabla f = 0$ at a point $(x^*, y^*)^T$ on the curve, then x^* will be a double zero of $f(x, h(x))$ since

$$\frac{\mathrm{d}}{\mathrm{d}x} f\bigl(x^*, h(x^*)\bigr) = \bigl[1, h'(x^*)\bigr]\nabla f = 0,$$

here $h(x)$ is the same as the function defined in the above paragraph. Therefore, $D(R, R_1, R_2)$-regular is a sufficient condition of the regularity.

4. Regular curve segments

In this section we characterize the regular curve segments defined on a triangle or a quadrilateral by the discriminating families introduced in Examples 3.1–3.4 in the last section. The goal is to arrive at the following statement:

For the BB-form polynomial F defined on a triangle or a quadrilateral and a specific discriminating family D, if certain conditions are satisfied, then $F = 0$ is D-regular. The “certain conditions” here will be the conditions imposed on the coefficients of F. For the purpose of practical use, we require that the validity of these conditions can be checked within finite steps of computation. The next theorem deals with the polynomials on the triangle and the D_1 discriminating family.

Theorem 4.1. *For the triangle* $[p_1 p_2 p_3]$, *let* $F_n(\alpha)$ *be defined as* (2.1) *and* $B_k(s) = \sum_{i+j=n-k} \beta_{ijk} B_j^{n-k}(s)$ (*see Fig.* 14(a)). *If there exists an integer* l $(0 < l < n)$ *such that* $B_k(s) \geqslant 0$ *for all* $s \in [0, 1]$ *and* $k < l$; $B_k(s) \leqslant 0$ *for all* $s \in [0, 1]$ *and* $k > l$ (*see Fig.* 15(a)) *and* $\sum_{k=0}^{l-1} B_k(s) > 0$, $\sum_{k=l+1}^{n} B_k(s) < 0$ *on* $(0, 1)$, *then the curve* $F_n(\alpha) = 0$ *is* $D_1([p_1 p_2 p_3], p_3, [p_1 p_2])$*-regular.*

The proof of the theorem is given in Appendix A.

Note that $B_k(s)$ depends on only the coefficients $\beta_{i,n-i-k,k}$ on the control points $\frac{i}{n} p_1 + \frac{n-i-k}{n} p_2 + \frac{k}{n} p_3$ for fixed k and $i = 0, \ldots, n-k$. These control points are on the line segment $[\frac{n-k}{n} p_2 + \frac{k}{n} p_3, \frac{n-k}{n} p_1 + \frac{k}{n} p_3]$ (see Fig. 14(a)). This comment applies to other theorems in this section, but with lines that are in different directions (see Fig. 14).

As an application of the theorem, we find that the curves in $D_2([p_1 p_2 p_3], p_2, p_3)$ are $D_1([p_1 p_2 p_3], p_1, [p_2 p_3])$-regular. Hence they are regular. Then it is easy to see D_2 is a discriminating family. Here the interesting thing is that starting with a naive discriminating

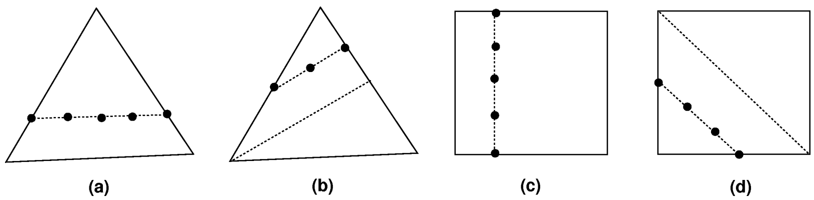

Fig. 14. The coefficients of $B_k(s)$ are on a line. (a) D_1-regular curve; (b) D_2-regular curve; (c) D_3-regular curve; (d) D_4-regular curve.

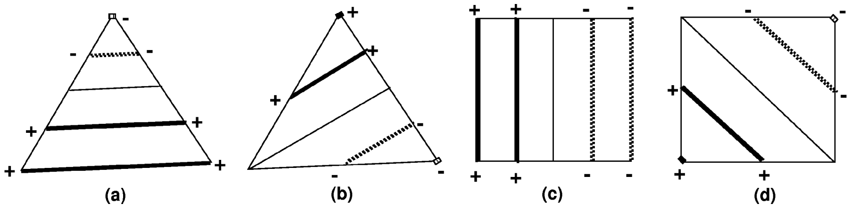

Fig. 15. The positive (real thick lines) and negative (dotted thick lines) $B_k(s)$. (a) D_1-regular curve; (b) D_2-regular curve; (c) D_3-regular curve; (d) D_4-regular curve.

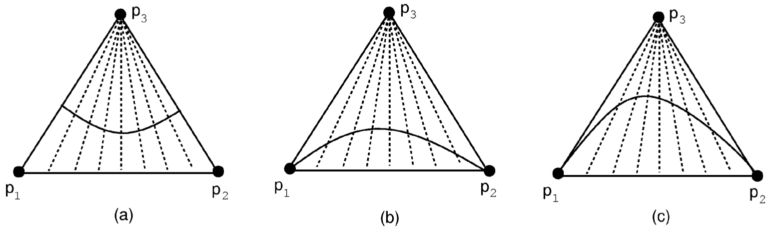

Fig. 16. The $D_1([p_1 p_2 p_3], p_3, [p_1 p_2])$-regular curves.

family, with the help of the corresponding characterization theorem we obtain other nontrivial discriminating families.

Note that a sufficient condition for a Bernstein polynomial $\sum_{i=0}^{n} \beta_i B_i^n(s)$ to be nonnegative on $[0, 1]$ is that the coefficients β_i are nonnegative. Hence we have the following corollary:

Corollary 4.1 (see Theorem 3.1 of (Bajaj and Xu, 1999)). *For the triangle $[p_1 p_2 p_3]$, let $F_n(\alpha)$ be defined as* (2.1). *If there exists an integer l $(0 < l < n)$ such that $\beta_{ijk} \geqslant 0$ for $k < l$ and $\beta_{ijk} \leqslant 0$ for $k > l$ and there is at least one strict inequality in each set of the inequalities, then the curve $F_n(\alpha) = 0$ is $D_1([p_1 p_2 p_3], p_3, [p_1 p_2])$-regular.*

The $D_1([p_1 p_2 p_3], p_3, [p_1 p_2])$-regular curves are between the point p_3 and the line segment $[p_1 p_2]$ and away from p_3 (if $\beta_{00n} \neq 0$) and the open line $(p_1 p_2)$ (see Fig. 16(a)). They can pass through p_1 and/or p_2 (see Fig. 16(b)) and furthermore, the curves can be tangent to the line $[p_1 p_3]$ and/or the line $[p_2 p_3]$ at p_1 and/or p_2 (see Fig. 16(c)).

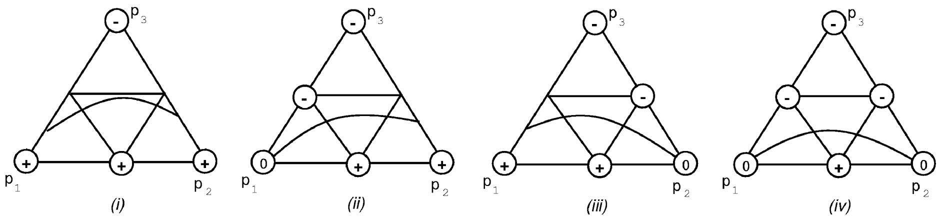

Fig. 17. The four cases in Theorem 4.2. The "+", "−" and "0" in the circles denote the corresponding coefficients are positive or non-negative, negative or non-positive and zero, respectively.

The conditions in Theorem 4.1 and Corollary 4.1 are easy to check and hence are very useful in constructing regular algebraic curves. However, these are not necessary conditions of regularity. Using Lemma 2.3, Lemma 2.4 and Lemma 2.5 we obtain necessary and sufficient conditions for regular quadratic and cubic curves.

Theorem 4.2. *Assume* $n = 2$ *and* $\beta_{002} < 0$. *Then* $F_2(\alpha) = 0$ *is* $D_1([p_1 p_2 p_3], p_3, [p_1 p_2])$-*regular if and only if one of the following conditions holds* (*see Fig.* 17):

- (i) $\beta_{200} > 0$, $\beta_{020} > 0$ *and* $\beta_{110} \geqslant 0$ *or* $\beta_{110}^2 < \beta_{200}\beta_{020}$ ($F_2 = 0$ *is between* p_3 *and* $[p_1 p_2]$).
- (ii) $\beta_{200} = 0$, $\beta_{020} > 0$, $\beta_{110} \geqslant 0$ *and* $\beta_{101} \leqslant 0$ ($F_2 = 0$ *passes through* p_1).
- (iii) $\beta_{200} > 0$, $\beta_{020} = 0$, $\beta_{110} \geqslant 0$ *and* $\beta_{011} \leqslant 0$ ($F_2 = 0$ *passes through* p_2).
- (iv) $\beta_{200} = \beta_{020} = 0$, $\beta_{110} > 0$, $\beta_{101} \leqslant 0$ *and* $\beta_{011} \leqslant 0$ ($F_2 = 0$ *passes through* p_1 *and* p_2).

Theorem 4.3. *Assume* $n = 3$ *and* $\beta_{003} < 0$. *Let* $S = \{s \in [0,1]\colon\ B_1(s) < 0,\ B_2(s) > 0\}$. *Then* $F_3(\alpha) = 0$ *is* $D_1([p_1 p_2 p_3], p_3, [p_1 p_2])$-*regular if and only if the following two conditions hold*:

$$B_0(s) > 0, \quad \forall s \in (0,1), \tag{4.1}$$

$$\begin{aligned} \Delta(s) := {} & B_0(s)^2 B_3(s)^2 - 3B_1(s)^2 B_2(s)^2 - 6B_0(s)B_1(s)B_2(s)B_3(s) \\ & + 4B_0(s)B_2(s)^3 + 4B_1(s)^3 B_3(s) > 0, \quad \forall s \in S. \end{aligned} \tag{4.2}$$

Note that conditions (4.1) and (4.2) in the theorem can be checked in a finite number of steps. Lemma 2.5 gives the necessary and sufficient conditions for $B_0(s) > 0$ on $(0,1)$. For condition (4.2), we first note that S is either an empty set or an interval or the union of two intervals, and S can be computed easily since B_1 and B_2 are polynomials of degree two and one, respectively. Hence (4.2) can be checked by Sturm's theorem. Let $\Delta_0(s), \Delta_1(s), \ldots, \Delta_6(s)$ be the Sturm sequence of $\Delta(s)$ and (a,b) be an interval of S. Then $\Delta(s) > 0$ on (a,b) if and only if the number of sign changes of the sequence $\Delta_0(a), \Delta_1(a), \ldots, \Delta_6(a)$ is the same as the number of sign changes of the sequence $\Delta_0(b), \Delta_1(b), \ldots, \Delta_6(b)$.

Now we are going to consider D_2-regular curves.

Theorem 4.4. *Let* $F_{2m}(\alpha)=\sum_{2i+j+k=2m}\beta_{2i,jk}B^{2m}_{2i,jk}(\alpha)$ *and*

$$B_n(s)=\sum_{j-k=2n}C^{2m}_{jk}/C^{2m}_{m-k,n+k}\beta_{2m-j-k,jk}B^{m+n}_{n+k}(s)$$

(*see Fig.* 14(b)). *If there exists an integer* l $(-m<l<m)$ *such that* $B_n(s)\leqslant 0$ *for all* $s\in[0,1]$ *and* $n<l$; $B_n(s)\geqslant 0$ *for all* $s\in[0,1]$ *and* $n>l$; *and* $\sum_{n=-m}^{l-1}B_n(s)<0$, $\sum_{n=l+1}^{m}B_n(s)>0$ *on* $(0,1)$ (*see Fig.* 15(b)), *then the curve* $F_{2m}(\alpha)=0$ *is* $D_2([p_1p_2p_3], p_2, p_3)$*-regular.*

The proof of the theorem is given in Appendix A.

Just as Corollary 4.1 is derived from Theorem 4.1, we have similar corollaries from Theorem 4.4 and Theorems 4.5, 4.8 below. We do not repeat them here.

Now we consider the algebraic curves defined on a quadrilateral $[p_1p_2p_3p_4]$. We shall characterize the coefficients β_{ij} such that the curve $G_{mn}(u,v)=0$ in the unit square S is a regular curve segment. This curve segment will be transformed to the given quadrilateral $[p_1p_2p_3p_4]$ by (2.2).

Theorem 4.5. *Let* $G_{mn}(u,v)$ *be defined as* (2.3) *and* $B_i(s)=\sum_{j=0}^{n}\beta_{ij}\ B^n_j(s)$ (*see Fig.* 14(c)). *If there exists an integer* l $(0<l<m)$ *such that* $B_i(s)\leqslant 0$ *for all* $s\in[0,1]$ *and* $0\leqslant i<l$; $B_i(s)\geqslant 0$ *for all* $s\in[0,1]$ *and* $l<i\leqslant m$ (*see Fig.* 15(c)) *and* $\sum_{i=0}^{l-1}B_i(s)<0$, $\sum_{i=l+1}^{m}B_i(s)>0$ *on* $(0,1)$, *then the curve* $G_{mn}(u,v)=0$ *is* $D_3([p_1p_2p_3p_4],[p_1p_2],[p_3p_4])$*-regular.*

The proof of the theorem is similar to the triangle case (see the proof of Theorem 4.1), and we omit it here.

Using Theorem 4.5, we can see that the curves in D_4 are D_3-regular. Hence it is easy to see that D_4 is a discriminating family.

The $D_3([p_1p_2p_3p_4],[p_1p_2],[p_3p_4])$-regular curves are between the line $[q_1q_2]$ and the line $[q_3q_4]$ (see Fig. 18(a)). They can pass through q_1 or q_2 or q_3 or q_4 (see Fig. 18(b)) and furthermore, the curves can be tangent to the edges at the vertices (see Figs. 18(c)–(d)). The above theorem implies that if the coefficients have a one-time sign change in the u or v direction, then the curve inside the unit square is regular.

Again, Theorem 4.5 gives sufficient conditions for $G_{mn}(u,v)=0$ to be $D_3([p_1p_2p_3p_4],[p_1p_2],[p_3p_4])$-regular. Now we give necessary and sufficient conditions for the case $m,n\leqslant 3$.

Theorem 4.6. *For* $m=2$ *and* $n\leqslant 3$, $G_{2,n}(u,v)=0$ *is* $D_3([p_1p_2p_3p_4],[p_1p_2],[p_3p_4])$*-regular if and only if*

$$\begin{aligned}&B_0(s)<0,\ B_2(s)>0,\quad \forall s\in(0,1),\\&\beta_{00}-\beta_{m0}<0,\quad \beta_{0n}-\beta_{mn}<0.\end{aligned}\tag{4.3}$$

Since $B_k(s)$ is a degree n polynomial, the lemmas in Section 2 give necessary and sufficient conditions for $B_0(s)<0$, $B_2(s)>0$.

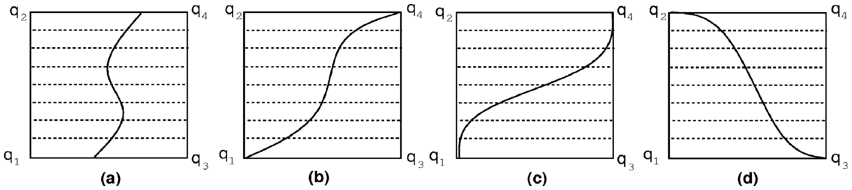

Fig. 18. The D_3-regular curves.

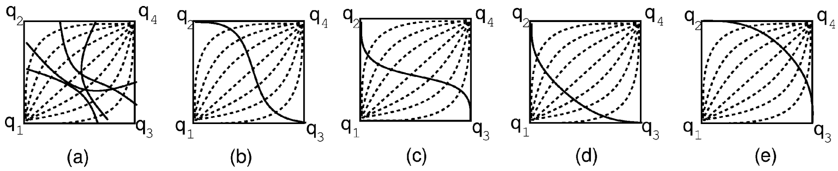

Fig. 19. The $D_4([p_1p_2p_3p_4], p_1, p_4)$-regular curves.

Theorem 4.7. *For $m = 3$ and $n \leqslant 3$, $G_{3,n}(u, v) = 0$ is $D_3([p_1p_2p_3p_4], [p_1\ p_2], [p_3p_4])$-regular if and only if* (4.3) *holds and*

$$\begin{aligned}\Delta(s) := {} & B_0(s)^2B_3(s)^2 - 3B_1(s)^2B_2(s)^2 - 6B_0(s)B_1(s)B_2(s)B_3(s)\\ & + 4B_0(s)B_2(s)^3 + 4B_1(s)^3B_3(s) > 0, \quad \forall s \in S,\end{aligned} \tag{4.4}$$

where $S = \{s \in [0, 1]\colon\ B_1(s) > 0,\ B_2(s) < 0\}$.

As in Theorem 4.3, condition (4.4) can be checked using Sturm's theorem.

Theorem 4.8. *Let $G_{mn}(u, v)$ be defined as* (2.3) *and $B_k(s) = \sum_{i=0}^{m} C_i^m C_{k-i}^n\ \beta_{i,k-i} B_{n+2i-k}^{m+n}(s)$ (see Fig.* 14(d)). *If there exists an integer l $(0 < l < m + n)$ such that $B_k(s) \leqslant 0$ for all $s \in [0, 1]$ and $0 \leqslant k < l$; $B_k(s) \geqslant 0$ for all $s \in [0, 1]$ and $l < k \leqslant m + n$ (see Fig.* 15(d)), *and $\sum_{k=0}^{l-1} B_k(s) < 0$, $\sum_{k=l+1}^{m+n} B_k(s) > 0$ on $(0, 1)$, then the curve $G_{mn}(u, v) = 0$ is $D_4([p_1p_2p_3p_4], p_1, p_4)$-regular in the unit square.*

The proof of the theorem is given in Appendix A.

The $D_4([p_1p_2p_3p_4], p_1, p_4)$-regular curves are between the point q_1 and point q_4 and away from q_1 and q_4 (see Fig. 19(a)). They can pass through q_2 or q_3 and can be tangent to the edges at the vertices (see Figs. 19(b)–(e)).

5. Display of regular algebraic curves

Displaying parametric curves is undoubtedly easier than displaying implicit algebraic curves. Here we show that fast display (graphing) algorithms exist for our regular curve families. For general degree curves these algorithms depend on a root finding routine for real univariate polynomial equations. If the degree of the polynomial is less than 5 (these are the most useful and important cases in CAGD), closed form solutions exist.

For the $D_1([p_1p_2p_3], p_3, [p_1p_2])$-regular curve defined by Theorem 4.1, we can evaluate the curve as follows: For a given $s \in [0, 1]$, determine t as in the proof of Theorem 4.1, then use (A.1) to compute $(\alpha_1(s,t), \alpha_2(s,t), \alpha_3(s,t))^{\mathrm{T}}$. To generate an ordered sequence of points on the curve for the computer graphics display, first choose a sequence $\{s_i\}$ $(0 = s_0 < s_1 < \cdots < s_l = 1)$, then compute $\{t_i\}$ and finally the points $\{(\alpha_1(s_i,t_i), \alpha_2(s_i,t_i), \alpha_3(s_i,t_i))^{\mathrm{T}}\}$. Connecting these points by lines yields a piecewise linear approximation of the curves. For the $D_2([p_1p_2p_3], p_2, p_3)$-regular curve defined by Theorem 4.4, formula (A.3) gives a closed form for evaluating $(\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$, where $s \in [0, 1]$ is given arbitrarily and t is determined as in the proof of Theorem 4.4. For the $D_4([p_1p_2p_3p_4], p_1, p_4)$-regular curve defined by Theorem 4.8, we use (A.5) to evaluate the curve. The display algorithms of the $D_3([p_1p_2p_3p_4], [p_1p_2], [p_3p_4])$-regular curves defined by Theorem 4.5 are similar.

6. Conclusions

We have introduced the concept of a discriminating family. Some concrete families are given. Using these families, we have characterized the bivariate BB-form polynomials on the triangle and the quadrilateral such that their zero contours are D-regular. The characterizations obtained are conditions imposed on the coefficients of the BB-form. The validity of these conditions can be checked within finite steps of computation. Furthermore, these discriminating families also serve as the tools to efficiently evaluate the D-regular curves for display.

Using these discriminating families, many regular curves are obtained. In addition to the D_1-regular curve family that has been used in an earlier paper (Bajaj and Xu, 1999), the D_2-regular, D_3-regular and D_4-regular curve families are newly introduced. The curves in each family have different features. The D_1-regular curves are always between p_3 and $[p_1p_2]$. The D_2-regular curves are between p_1 and p_2. The D_3-regular curves are between $[p_1p_2]$ and $[p_3p_4]$, while the D_4-regular curves are between p_1 and p_4. These different features may serve different purposes in the diverse applications of CAGD.

In the literature, many attempts have been made in using algebraic curves defined on a triangle. The introduction of the D_3-regular and D_4-regular curve families provides the capability for using algebraic curves defined on a quadrilateral.

In part II and part III of this trilogy of papers, we will exploit the use of these curve families in interpolation, approximation and scattered data fitting.

Appendix A

The proof of Theorem 4.1. For a given $s \in [0, 1]$, let

$$\begin{cases} s = \alpha_2/(\alpha_1 + \alpha_2), \\ t = \alpha_3. \end{cases}$$

That is,

$$\begin{cases} \alpha_1 = (1-t)(1-s), \\ \alpha_2 = (1-t)s, \\ \alpha_3 = t. \end{cases} \tag{A.1}$$

Then

$$F_n(\alpha) = \sum_{k=0}^{n} \sum_{i+j=n-k} \beta_{ijk} \frac{n!}{i!j!k!} \alpha_1^i \alpha_2^j \alpha_3^k = \sum_{k=0}^{n} B_k^n(t) \sum_{i+j=n-k} \beta_{ijk} B_j^{n-k}(s)$$
$$= \sum_{k=0}^{n} B_k(s) B_k^n(t).$$

For a given s, since $B_k(s) \geqslant 0$ for $k < l$, $B_k(s) \leqslant 0$ for $k > l$ and there is a strict inequality in each of them, $F_n(\alpha) = 0$ has exactly one root $t \in (0,1)$ by Lemma 2.1. This $(s,t)^{\mathrm{T}}$ gives us a unique α by (A.1). □

The proof of Theorem 4.4. The case $s = 0$, when the curve in $D_2([p_1 p_2 \ p_3], p_2, p_3)$ degenerates to straight lines $\alpha_2 = 0$ and $\alpha_3 = 0$, needs to be considered separately. For instance, if the coefficients $\beta_{2i,0,2m-2i}$, $i = 0, 1, \ldots, m$, have a one-time sign change, we let $\alpha_2 = 0$, and then

$$F_{2m}(\alpha) = \sum_{2i+k=2m} \beta_{2i,0,k} B_{2i,0,k}^{2m}(\alpha_1, 0, 1-\alpha_1) = \sum_{i=0}^{m} \beta_{2i,0,2m-2i} B_{2i}^{2m}(\alpha_1).$$

Hence the equation $F_{2m}(\alpha) = F_{2m}(\alpha_1, 0, 1-\alpha_1) = 0$ with α_1 as the unknown has one root in $[0,1]$. The case that $\beta_{2i,2m-2i,0}$, $i = 0, 1, \ldots, m$, has a one-time sign change is similar. Now suppose $s \in (0,1]$. Let

$$\begin{cases} s = \alpha_2\alpha_3/(\alpha_1^2 + \alpha_2\alpha_3), \\ t = \alpha_3/(\alpha_2 + \alpha_3). \end{cases} \tag{A.2}$$

Then it follows from the second equality of (A.2) that $\alpha_2 = (1-\alpha_1)(1-t)$, $\alpha_3 = t(1-\alpha_1)$. Substituting these into the first equality of (A.2), we have $(1-s)(1-t)t(1-\alpha_1)^2 = s\alpha_1^2$, from which we get

$$\begin{cases} \alpha_1 = \dfrac{\sqrt{(1-s)(1-t)t}}{\sqrt{s} + \sqrt{(1-s)(1-t)t}}, \\[2ex] \alpha_2 = \dfrac{\sqrt{s(1-t)^2}}{\sqrt{s} + \sqrt{(1-s)(1-t)t}}, \\[2ex] \alpha_3 = \dfrac{\sqrt{st^2}}{\sqrt{s} + \sqrt{(1-s)(1-t)t}}. \end{cases} \tag{A.3}$$

For $j - k = 2n$ (that is $j + k = 2n + 2k$),

$$B_{2m-j-k,jk}^{2m}(\alpha) = C_{jk}^{2m} \alpha_1^{2(m-n-k)} \alpha_2^j \alpha_3^k$$
$$= C_{jk}^{2m} \frac{s^{n+k}(1-s)^{m-n-k} t^{m-n} (1-t)^{m+n}}{(\sqrt{s} + \sqrt{(1-s)(1-t)t})^{2m}}$$
$$= \frac{C_{jk}^{2m}}{C_{m-k,n+k}^{2m}} \frac{B_{n+k}^{m+n}(s) B_{m-n}^{2m}(t)}{(\sqrt{s} + \sqrt{(1-s)(1-t)t})^{2m}}.$$

Then

$$
\begin{aligned}
F_{2m}(\alpha) &= \sum_{n=-m}^{m} \sum_{j-k=2n} \beta_{2m-j-k,\,jk} B^{2m}_{2m-j-k,\,jk}(\alpha) \\
&= \frac{1}{(\sqrt{s}+\sqrt{(1-s)(1-t)t})^{2m}} \sum_{n=-m}^{m} B_n(s) B^{2m}_{m-n}(t).
\end{aligned}
$$

It follows from the assumption of the theorem that, for a given s, $B_n(s) \leqslant 0$ for $n < l$, and $B_n(s) \geqslant 0$ for $n > l$, and there is a strict inequality in each of them. Then by Lemma 2.1 $F_{2m}(\alpha) = 0$ has exactly one root $t \in (0,1)$ for any given $s \in (0,1]$. This $(s,t)^{\mathrm{T}}$ gives us a unique $(\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$ by (A.3). □

The proof of Theorem 4.8. We claim that every member in the family of hyperbola will intersect the curve $G_{mn}(u,v) = 0$ once and only once. When $s = 0$ or $s = 1$, the curve in that family degenerates to the boundary of the unit square, and our claim is trivially true. Now we suppose $s \in (0,1)$. Let

$$
\begin{cases}
s = \dfrac{u(1-v)}{u(1-v)+(1-u)v}, \\
t = \dfrac{uv}{uv+(1-u)(1-v)},
\end{cases}
\tag{A.4}
$$

where $t \in (0,1)$. It is not difficult to show, from (A.4), that

$$
\begin{cases}
u = \dfrac{\sqrt{st}}{\sqrt{st}+\sqrt{(1-s)(1-t)}}, \\
v = \dfrac{\sqrt{(1-s)t}}{\sqrt{(1-s)t}+\sqrt{s(1-t)}}.
\end{cases}
\tag{A.5}
$$

Hence

$$
\begin{aligned}
G_{mn}(u,v) &= \sum_{i=0}^{m} \sum_{j=0}^{n} \beta_{ij} B_i^m(u) B_j^n(v) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{\beta_{ij} C_i^m C_j^n \sqrt{B^{m+n}_{n+i-j}(s)}\sqrt{B^{m+n}_{i+j}(t)}}{\sqrt{C^{m+n}_{i+j} C^{m+n}_{n+i-j}}\ \mu(s,t)^m\ \nu(s,t)^n} \\
&= \sum_{k=0}^{m+n} \sum_{i=0}^{m} \frac{C_i^m C_{k-i}^n \beta_{i,k-i} \sqrt{B^{m+n}_{n+2i-k}(s)}\sqrt{B^{m+n}_{k}(t)}}{\sqrt{C^{m+n}_{k} C^{m+n}_{n+2i-k}}\ \mu(s,t)^m\ \nu(s,t)^n} \\
&= \frac{1}{\mu(s,t)^m\ \nu(s,t)^n} \sum_{k=0}^{m+n} \beta_k(s) \sqrt{B_k^{m+n}(t)},
\end{aligned}
$$

where $\mu(s,t) = \sqrt{st} + \sqrt{(1-s)(1-t)}$, $\nu(s,t) = \sqrt{(1-s)t} + \sqrt{s(1-t)}$,

$$
\beta_k(s) = \sum_{i=0}^{m} C_i^m C_{k-i}^n \beta_{i,k-i} \sqrt{B^{m+n}_{n+2i-k}(s)} \Big/ \sqrt{C_k^{m+n} C^{m+n}_{n+2i-k}},
$$

and $\beta_{ij} = 0$ for $j < 0$ or $j > n$. Hence $G_{mn}(u, v) = 0$ is equivalent to

$$\sum_{k=0}^{m+n} \beta_k(s)\sqrt{B_k^{m+n}(t)} = 0.$$

Under the assumptions of this theorem and Lemma 2.2, $\beta_k(s) \leqslant 0$ for $k = 0, 1, \ldots, l-1$, $\beta_k(s) \geqslant 0$ for $k = l+1, \ldots, m+n$ there exist strict inequalities in each set of the inequalities. It follows from Lemma 2.1 that $G_{mn}(u, v) = 0$ has a single root t in (0, 1) for any given $s \in (0, 1)$. Hence the hyperbolic in D_4 has only one intersection with the curve $G_{mn}(u, v) = 0$ in the unit square. The intersection point is defined by (A.5). Therefore the curve $G_{mn}(u, v) = 0$ in the unit square is regular. □

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