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NURBS Approximation of A-Splines and A-Patches

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ABSTRACT

Given A-spline curves and A-patch surfaces that are implicitly defined on triangles and tetrahedra, we determine their NURBS representations. We provide a trimmed NURBS form for A-spline curves and a parametric tensor-product NURBS form for A-patch surfaces. We concentrate on cubic A-patches, providing a C^1-continuous surface that interpolates a given triangulation together with surface normals at the vertices. In many cases we can generate cubic trimming curves that are rationally parametrizable on the triangular faces of the tetrahedra; for the remaining faces we resort to using quadratic curves, which are always rationally parametrizable, to approximate the cubic trimming curves.

1. Introduction

Low degree polynomial or algebraic surfaces can often have dual parametric and implicit representations. Each form has its distinct advantages. The parametric polynomial spline in B-spline Basis (B) and Bernstein-Bézier (BB) bases are currently overwhelmingly popular in commercial and industrial CAGD systems. In this paper we show how to generate trimmed, parametric B-spline and BB-spline representations for a collection of implicitly defined algebraic surface patches introduced in Refs. 1–3. Each implicit algebraic surface patch (A-patch) is a smooth,

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bounded, zero-contour of a trivariate polynomial, defined within a tetrahedron for the barycentric B/BB basis and within a box for the tensor product B/BB basis (see Ref. 4, Chap. 4 for details). We also show how to convert the trimming curves of the input A-patch collection into rational parametric form in the same basis as the surface conversion, yielding standard trimmed NURBS representations [1]. As NURBS representations are efficient to compute and are a very common standard form for splines, being able to represent A-patches as NURBS is highly desirable.

Many low degree implicit curves or surfaces are rational, i. e., convertible into rational parametric form. All degree two curves (conics) are rational, but only the subset of singular degree three (cubics) are rational, i. e. elliptic cubics are non-singular and not rationally parametrizable [2]. In general, a necessary and sufficient condition for the global rationality of an algebraic curve of arbitrary degree is given by the Cayley-Reimann criterion: *a curve is rational if and only if* $g = 0$, where g, the genus of the curve, is a measure of the deficiency of the curves' singularities from its maximum allowable limit. For surfaces, all implicit quadratic and cubic surfaces can be rationally parametrizable (except the elliptic cubic cylinders or cones). A method for rationally parametrizing general quadratic curves and surfaces is given in Refs. 7 and 8. These are all we need to rationally parametrize C^0 quadratic A-patches. Similarly, techniques for parametrizing rational cubic curves and surfaces have previously been given in Refs. 9–12. A proper subset of higher degree surface can be rationally parametrized, with a necessary and sufficient criterion given due to Castelnuovo [3]. Since it is not always possible to perform exact conversions to rational parametric form, we appeal to approximate conversions when necessary. However, we preserve the continuity of the spline surface to be converted, that is, we construct trimmed NURBS representations of C^1 cubic A-splines and C^1 cubic A-patches.

The rest of the paper is organized as follows. Section 2 discusses the conversion of A-splines curves, which are also the boundary (trimming) curves of A-patches, given in implicit form to NURBS representation. In Section 3.2 we first classify the cases of exact convertibility of C^1 cubic A-patch splines into trimmed NURBS form. When exact convertibility is not possible, we show how to generate "fair" approximate trimmed NURBS. Section 4 concludes the paper. Details of the derivations and examples are presented in the Appendices.

2. NURBS Representation of A-splines

An A-spline of degree n over the triangle $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ is defined by

$$G_n(x,y) := F_n(\boldsymbol{\alpha}) = F_n(\alpha_1,\alpha_2,\alpha_3) = 0, \tag{1}$$

where

$$F_n(\alpha_1,\alpha_2,\alpha_3) = \sum_{i+j+k=n} b_{ijk} B^n_{ijk}(\alpha_1,\alpha_2,\alpha_3), \quad B^n_{ijk}(\alpha_1,\alpha_2,\alpha_3) = \frac{n!}{i!j!k!}\alpha_1^i\alpha_2^j\alpha_3^k,$$

and $(x, y)^T$ and $(\alpha_1, \alpha_2, \alpha_3)^T$ are related by

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}. \tag{2}$$

Here the objective is to get an A-spline parametrization of the following form:

$$\mathbf{X}(t) = \sum_{i=0}^{n} w_i B_i^n(t)\mathbf{b}_i \Big/ \sum_{i=0}^{n} w_i B_i^n(t), \qquad t \in [0,1], \tag{3}$$

where $\mathbf{b}_i \in I\!\!R^3$, $w_i \in I\!\!R$, and $B_i^n(t) = \{n!/[i!(n-i)!]\}t^i(1-t)^{n-i}$. Without loss of generality, we may assume that $w_0 = 1$ (otherwise we could divide through by t and have a parametrization of one lower degree). Next, under the transformation

$$t = \frac{t' + at'}{1 + at'}, \quad a > -1, \;\; t' \in [0,1], \tag{4}$$

the curve (3) will preserve its form, that is

$$\mathbf{X}(t) = \sum_{i=0}^{n}(1+a)^i w_i B_i^n(t')\mathbf{b}_i \Big/ \sum_{i=0}^{n}(1+a)^i w_i B_i^n(t'), \qquad t' \in [0,1].$$

Therefore, we may assume further that $w_n = 1$ by setting $a = w_n^{-1/n} - 1$, which makes $(1+a)^n w_n = 1$, in the transformation (4).

We consider first convex C^1 continuous A-splines (see Fig. 1(a)). An A-spline being C^1 implies that $b_{n00} = b_{0n0} = b_{n-1,01} = b_{0,n-1,1} = 0$ [4]. The C^0 continuous A-splines on the triangle $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ can be made into C^1 continuous A-splines on the triangle $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3']$ (see Fig. 1(c)) through the use of the subdivision formula [5]. In our applications in the parametrization of cubic ($n = 3$) A-patches, the coefficients $a,\ b,\ c$ are fixed and $d,\ e,\ f$ are parameters to be determined, where

$$a = b_{210}, \quad b = b_{120}, \quad c = b_{111}, \quad d = b_{102}, \quad e = b_{012}, \quad f = b_{003}.$$

The non-convex case (see Fig. 1(b)) can be converted to the convex case by first computing the intersection point $\mathbf{p}_2'$, which leads to a linear equation for $n = 3$, and then computing the tangent of the curve at $\mathbf{p}_2'$. Note that this tangent does not depend upon the coefficients $d,\ e,\ f$.

2.1. Quadratic A-splines

It is not difficult to see that the parametric form of a C^1-continuous quadratic A-spline should have the following form (see Fig. 1(d)) since it interpolates the points $\mathbf{p}_1$ and $\mathbf{p}_2$ and is tangent to the lines $[\mathbf{p}_1\mathbf{p}_3]$ and $[\mathbf{p}_2\mathbf{p}_3]$ at the points $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively.

$$\mathbf{X}(t) = \frac{\mathbf{p}_1 B_0^2(t) + w_1\mathbf{p}_3 B_1^2(t) + \mathbf{p}_2 B_2^2(t)}{B_0^2(t) + w_1 B_1^2(t) + B_2^2(t)}, \quad t \in [0,1], \tag{5}$$

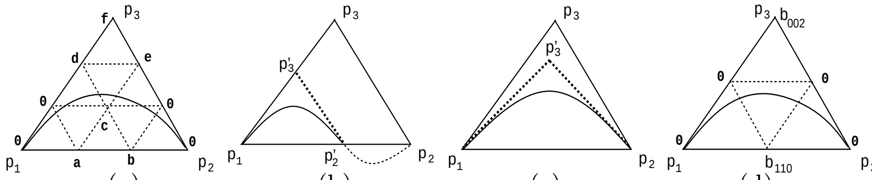

Fig. 1: (a): Convex case; (b) Non-convex case; (c) C^0 A-spline; (d) Quadratic A-spline.

where w_1 is a parameter to be determined. This is called a (2/2) rational parametri-lb zation because the of the numerator and denominator are each of degree 2 in t. We show in Appendix 1 that

$$w_1 = \sqrt{-\frac{b_{110}}{2b_{002}}} \geq 0. \tag{6}$$

2.2. Cubic A-splines

We first show that an irreducible C^1-continuous cubic A-spline never has a (2/2) rational parametrization. If we substitute the α's defined by (A.1) into $F_3(\boldsymbol{\alpha}) = 0$, we have $\sum_{i=0}^{6} c_i B_i^6(t) \equiv 0$, where

$$c_0 = b_{300}, \quad c_1 = b_{201}w_1, \quad c_2 = \frac{1}{5}b_{210} + \frac{4}{5}b_{102}w_1^2, \quad c_3 = \frac{3}{5}b_{111}w_1 + \frac{2}{5}b_{003}w_1^3,$$
$$c_6 = b_{030}, \quad c_5 = b_{021}w_1, \quad c_4 = \frac{1}{5}b_{120} + \frac{4}{5}b_{012}w_1^2.$$

Since $B_i^6(t),\ i = 0,\ \ldots,\ 6$, are linearly independent, we have $c_i = 0,\ \ i = 0,\ \ldots,\ 6$. It then follows that

$$a + 4dw_1^2 = 0, \quad 3cw_1 + 2fw_1^3 = 0, \quad b + 4ew_1^2 = 0$$

and hence $w_1 = \sqrt{-a/4d}$. The coefficients of the A-spline must satisfy

$$\frac{d}{a} = \frac{f}{6c} = \frac{e}{b}, \tag{7}$$

where $a = b_{210},\ \ b = b_{120},\ \ c = b_{111},\ \ d = b_{102},\ \ e = b_{012},\ \ f = b_{003}$. However, the substitutions (7) turn the A-spline $F_3(\boldsymbol{\alpha}) = 3a\alpha_1^2\alpha_2 + 3b\alpha_1\alpha_2^2 + 6c\alpha_1\alpha_2\alpha_3 + 3d\alpha_1\alpha_3^2 + 3e\alpha_2\alpha_3^2 + f\alpha_3^3 = 0$ into $F_3(\boldsymbol{\alpha}) = (\alpha_1\alpha_2 + d/a\,\alpha_3^2)(a\alpha_1 + b\alpha_2 + 2c\alpha_3) = 0$, which is the product of a line and an ellipse. The parametrization covers the ellipse, and is essentially the same as the (2/2) parametrization of a quadratic A-spline.

The (3/3) rational parametric form of a C^1-continuous cubic A-spline should have the following form in order to interpolate the points $\mathbf{p}_1$ and $\mathbf{p}_2$ and be tangent to the lines $[\mathbf{p}_1\mathbf{p}_3]$ and $[\mathbf{p}_2\mathbf{p}_3]$ at $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively:

$$\mathbf{X}(t) = \frac{\mathbf{p}_1 B_0^3(t) + w_1[\mathbf{p}_1 + \alpha(\mathbf{p}_3 - \mathbf{p}_1)]B_1^3(t) + w_2[\mathbf{p}_2 + \beta(\mathbf{p}_3 - \mathbf{p}_2)]B_2^3(t) + \mathbf{p}_2 B_3^3(t)}{B_0^3(t) + w_1 B_1^3(t) + w_2 B_2^3(t) + B_3^3(t)}, \tag{8}$$

where α, β, w_1, w_2 are parameters to be determined.

We will show that the equation

$$\begin{aligned}
G_{[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]}(a,b,c,d,e,f) &= 48a^3e^3f^2 - 9a^2b^2f^4 + 72a^2bcef^3 - 72a^2bde^2f^2 \\
&- 96a^2c^2e^2f^2 - 288a^2cde^3f + 432a^2d^2e^4 + 72ab^2cdf^3 - 72ab^2d^2ef^2 \\
&- 8abc^3f^3 - 552abc^2def^2 + 1152abcd^2e^2f - 864abd^3e^3 + 48ac^4ef^2 \\
&+ 576ac^3de^2f - 864ac^2d^2e^3 + 48b^3d^3f^2 - 96b^2c^2d^2f^2 - 288b^2cd^3ef \\
&+ 432b^2d^4e^2 + 48bc^4df^2 + 576bc^3d^2ef - 864bc^2d^3e^2 \\
&- 288c^5def + 432c^4d^2e^2 = 0 \qquad (9)
\end{aligned}$$

gives a condition on the A-spline coefficients that guarantee the A-spline has a rational parametrization. The proof of this is rather technical and is given in Appendix B.

We will wish to construct rationally parametrizable cubic A-splines defined on a triangle $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ and passing through $\mathbf{p}_1$ and $\mathbf{p}_2$, that are not necessarily tangent to the edges $[\mathbf{p}_1\mathbf{p}_3]$ and $[\mathbf{p}_2\mathbf{p}_3]$ at $\mathbf{p}_1$ and $\mathbf{p}_2$. This situation is illustrated in Fig. 1(c), where the tangent lines at $\mathbf{p}_1$ and $\mathbf{p}_2$ intersect at some other point $\mathbf{p}_3'$. These cubic A-splines will have one degree of freedom, the weight b_{003}, which we will use to satisfy (9). In order to accomplish this we define a coordinate system $\alpha_1'\alpha_2'\alpha_3'$ (where $\alpha_1' + \alpha_2' + \alpha_3' = 1$) that has its origin $(0,0,1)$ at $\mathbf{p}_3'$ instead of $\mathbf{p}_3$, while keeping the points $(1,0,0)$ and $(0,1,0)$ fixed.

The general cubic curve passing through $\mathbf{p}_1$ and $\mathbf{p}_2$ is

$$\begin{aligned}
&3b_{210}\alpha_1^2\alpha_2 + 3b_{201}\alpha_1^2\alpha_3 + 3b_{120}\alpha_1\alpha_2^2 + 6b_{111}\alpha_1\alpha_2\alpha_3 \\
&\quad + 3b_{102}\alpha_1\alpha_3^2 + 3b_{021}\alpha_2^2\alpha_3 + 3b_{012}\alpha_2\alpha_3^2 + b_{003}\alpha_3^3 = 0 \; . \qquad (10)
\end{aligned}$$

The tangent lines to this curve at $\mathbf{p}_1$ and $\mathbf{p}_2$ are

$$b_{210}\alpha_2 + b_{201}\alpha_3 = 0, \qquad\qquad b_{120}\alpha_1 + b_{021}\alpha_3 = 0 \;\; ,$$

and these intersect at the point

$$(\alpha_1, \alpha_2, \alpha_3) = \frac{(b_{210}b_{021}, b_{201}b_{120}, -b_{210}b_{120})}{b_{210}b_{021} + b_{201}b_{120} - b_{210}b_{120}} \;\; .$$

The linear transformation that maps $(\alpha_1, \alpha_2, \alpha_3) = (1,0,0), (0,1,0), (b_{210}b_{021}, b_{201}b_{120}, -b_{210}b_{120})/(b_{210}b_{021} + b_{201}b_{120} - b_{210}b_{120})$ into $(\alpha_1', \alpha_2', \alpha_3') = (1,0,0), (0,1,0), (0,0,1)$, respectively is

$$\begin{aligned}
\alpha_1 &= \alpha_1' + \frac{b_{210}b_{021}}{b_{210}b_{021} + b_{201}b_{120} - b_{210}b_{120}}\,\alpha_3' \\
\alpha_2 &= \alpha_2' + \frac{b_{201}b_{120}}{b_{210}b_{021} + b_{201}b_{120} - b_{210}b_{120}}\,\alpha_3' \\
\alpha_3 &= -\frac{b_{210}b_{120}}{b_{210}b_{021} + b_{201}b_{120} - b_{210}b_{120}}\,\alpha_3' \qquad (11)
\end{aligned}$$

with the inverse

$$\alpha_1' = \alpha_1 - \frac{b_{021}}{b_{120}}\,\alpha_3, \quad \alpha_2' = \alpha_2 - \frac{b_{201}}{b_{210}}\,\alpha_3,$$
$$\alpha_3' = -\frac{b_{210}b_{021} + b_{201}b_{120} - b_{210}b_{120}}{b_{210}b_{120}}\,\alpha_3\ . \tag{12}$$

Thus the transformation (11) maps (10) into an equation of the form

$$a'{\alpha_1'}^2\alpha_2' + 3b'\alpha_1'{\alpha_2'}^2 + 6c'\alpha_1'\alpha_2'\alpha_3' + 3d'\alpha_1'{\alpha_3'}^2 + 3e'\alpha_2'{\alpha_3'}^2 + f'{\alpha_3'}^3 = 0\ .$$

An example of this being used to construct a rational trimming curve is given in Example 1 in Appendix D, for the face $[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3]$.

3. NURBS Representation of A-patches

An A-patch of degree n over the tetrahedron $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ is defined by

$$G_n(x,y,z) := F_n(\boldsymbol{\alpha}) = F_n(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = 0, \tag{1}$$

where

$$F_n(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \sum_{i+j+k+l=n} a_{ijkl}B^n_{ijkl}(\alpha_1,\alpha_2,\alpha_3,\alpha_4), \tag{2}$$

$$B^n_{ijkl}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \frac{n!}{i!j!k!l!}\alpha_1^i\alpha_2^j\alpha_3^k\alpha_4^l,$$

and $(x,y,z)^T$ and $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^T$ are related by

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}. \tag{3}$$

3.1. Quadratic A-patches

The construction details of quadratic A-patches can be found in Ref. 4. The derivation of the parametrization of quadratic curves and surfaces is given in Ref. 7 and the BB form is given in Ref. 8. Details of the parametrization for the trimming curves, which are all quadratic, have been presented in Section 2.1. For brevity we will not repeat all of these conversion formulas here.

3.2. Cubic A-patches

The construction details of cubic A-patches can be found in Refs. 3 and 4. Appendix C summarizes all the required computation formulas for BB-form coefficients of the A-patches for four adjacent face tetrahedra (see Fig. C.1) and six edge tetrahedra (see Fig. C.2). With all these computational formulas, there are still several degrees of freedom. Specifically, the weights a^m_{1110}, a^m_{1002}, a^m_{0102}, a^m_{0012}, a^m_{0003}, and b^m_{2001} may be chosen freely. We wish to use these degrees of freedom to

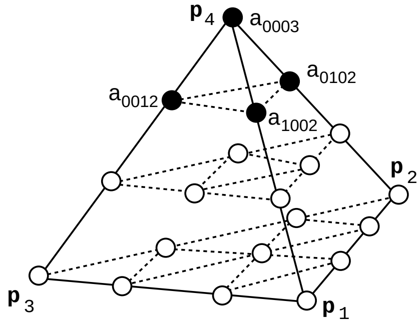

Fig. 2: Four free weights of a cubic A-patch for a face tetrahedron

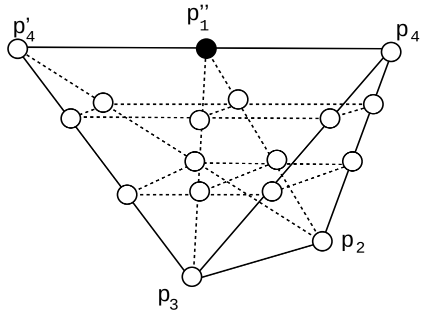

Fig. 3: One free weight of a cubic A-patch for an edge tetrahedron

make the cubic A-patch single-sheeted and have boundary curves that are rational parametric.

3.2.1. Rational parametric boundary curves

For the face A-patch, we have four weights free [6] (see Fig. 2). These weights will be used to make the three boundary curves rationally parametrizable. Since forcing a C^1-continuous cubic A-spline to be rationally parametrizable requires the imposition of a single constraint (9), the splines on the three faces $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_4]$, $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$, and $[\mathbf{p}_3\mathbf{p}_1\mathbf{p}_4]$ lead to three equations:

$$\begin{array}{rcl} G_{[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_4]}(a_{2100}, a_{1200}, a_{1101}, a_{1002}, a_{0102}, a_{0003}) & = & 0 \\ G_{[\mathbf{p}_1\mathbf{p}_3\mathbf{p}_4]}(a_{1020}, a_{2010}, a_{1011}, a_{0012}, a_{1002}, a_{0003}) & = & 0 \\ G_{[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]}(a_{0210}, a_{0120}, a_{0111}, a_{0102}, a_{0012}, a_{0003}) & = & 0 \ , \end{array} \quad (4)$$

where $G_{[\mathbf{p}_i\mathbf{p}_j\mathbf{p}_k]}(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ is defined by (9), and four unknowns $(a_{1002}, a_{0102}, a_{0012}, a_{0003})$.

For the edge patch, we have one weight free on the interface $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1'']$ (see Fig. 3). If we let b_{ijkl} denote the weights for tetrahedron $[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ (as in Appendix C), then the free weight is b_{3000}. Solving the equation $G_{[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1'']}(b_{0210}, b_{0120}, b_{1110}, b_{2100}, b_{2010}, b_{3000}) = 0$, provides the required coefficient.

If we are given two rationally parametrized curves on a cubic surface, we can obtain a rational parametrization for the surface in a manner similar to that in Ref. 12. The idea is that a line that passes through two nonsingular real points on a cubic surface must intersect the surface in a third real point. Let the two curves on the surface $f(x,y,z) = 0$ be

$$\mathbf{c}_1(u) = [x_1(u) \ \ y_1(u) \ \ z_1(u)]^T \text{ and } \mathbf{c}_2(u) = [x_2(u) \ \ y_2(u) \ \ z_2(u)]^T \ .$$

Then the cubic parametrization formula for a point $\mathbf{p}(u,v)$ on the surface is

$$\mathbf{p}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} = \frac{a\mathbf{c}_1 + b\mathbf{c}_2}{a+b} = \frac{a(u,v)\mathbf{c}_1(u) + b(u,v)\mathbf{c}_2(v)}{a(u,v) + b(u,v)} \quad (5)$$

where

$$
\begin{array}{rclcl}
a & = & a(u,v) & = & \nabla f(\mathbf{c}_2(v)) \cdot [\mathbf{c}_1(u) - \mathbf{c}_2(v)] \\
b & = & b(u,v) & = & \nabla f(\mathbf{c}_1(u)) \cdot [\mathbf{c}_1(u) - \mathbf{c}_2(v)] \ .
\end{array}
$$

A simpler, lower degree parametrization can be obtained if we know and can use two skew lines on the cubic surface rather than cubic curves. This was the approach in Ref. 12, and results in a 1-to-1 covering of the cubic surface, while using cubic curves as $\mathbf{c}_1$ and $\mathbf{c}_2$ can result in a 9-to-1 covering, which means that there are nine values of the ordered pair (u,v) that map to almost all points on the surface. Nonsingular cubic surfaces can be put into five categories based on the number of real lines upon them, and rational parametrizations are possible in four of them. Examples of these rational parametrizations are given in Appendix D.

3.2.2. Addition of a singular point

In this section we determine the free coefficients (dropping the superscript ^{1}) a_{1002}, a_{0102}, a_{0012}, and a_{0003} of tetrahedron $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ by forcing the cubic surface to have a singular point at a specific location outside the tetrahedron, say at $\mathbf{p}_0$: $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-k, -k, -k, 3k+1)$ for some $k \geq 0$. A singular point on the surface $S(\boldsymbol{\alpha}) = 0$ is one where the gradient vanishes, so that $\nabla S(\mathbf{p}_0) = \mathbf{0}$. Here S is considered to be a function of the three independent variables $\{\alpha_1, \alpha_2, \alpha_3\}$. The conditions that $S(-k,-k,-k) = 0$ and $(\partial S/\partial \alpha_i)_{\boldsymbol{\alpha}=\mathbf{p}_0} = 0$, $i = 1,2,3$, are equivalent to

$$
\begin{array}{rcl}
0 & = & -3k^3(a_{2100} + a_{2010} + a_{1200} + 2a_{1110} + a_{1020} + a_{0210} + a_{0120}) \\
& & + 3k^2(3k+1)(a_{2001} + 2a_{1101} + 2a_{1011} + a_{0201} + 2a_{0111} + a_{0021}) \\
& & - 3k(3k+1)^2(a_{1002} + a_{0102} + a_{0012}) + (3k+1)^3 a_{0003}, \\
0 & = & k^2(2a_{2100} + 2a_{2010} + a_{1200} + 2a_{1110} + a_{1020} - a_{0201} - 2a_{0111} - a_{0021}) \\
& & - k(7k+2)a_{2001} - 2k(4k+1)(a_{1101} + a_{1011}) + 2k(3k+1)(a_{0102} + a_{0012}) \\
& & + (5k+1)(3k+1)a_{1002} - (3k+1)^2 a_{0003}, \\
0 & = & k^2(a_{2100} - a_{2001} + 2a_{1200} + 2a_{1110} - 2a_{1011} + 2a_{0210} + a_{0120} - a_{0021}) \\
& & - k(7k+2)a_{0201} - 2k(4k+1)(a_{1101} - a_{0111}) + 2k(3k+1)(a_{1002} + 8a_{0012}) \\
& & + (5k+1)(3k+1)a_{0102} - (3k+1)^2 a_{0003}, \\
0 & = & k^2(a_{2010} - a_{2001} + 2a_{1110} - 2a_{1101} + 2a_{1020} + a_{0210} - a_{0201} + 2a_{0120}) \\
& & - k(7k+2)a_{0021} - 2k(4k+1)(a_{1011} + a_{0111}) + 2k(3k+1)(a_{1002} + 8a_{0102}) \\
& & + (5k+1)(3k+1) + a_{0012} - (3k+1)^2 a_{0003},
\end{array}
$$

and this system has the solution

$$
\begin{aligned}
a_{1002} &= k[-(2a_{2100} + 2a_{2010} + a_{1200} + 2a_{1110} + a_{1020})k \\
&\quad + 2(3k+1)(a_{2001} + a_{1101} + a_{1011})]/(3k+1)^2, \\
a_{0102} &= k[-(a_{2100} + 2a_{1200} + 2a_{1110} + 2a_{0210} + a_{0120})k \\
&\quad + 2(3k+1)(a_{1101} + a_{0201} + a_{0111})]/(3k+1)^2, \\
a_{0012} &= k[-(a_{2010} + 2a_{1110} + 2a_{1020} + a_{0210} + 2a_{0120})k
\end{aligned}
$$

$$+ \ 2(3k+1)(a_{1011}+a_{0111}+a_{0021})]/(3k+1)^2,$$

$$a_{0003} = 3k^2[-2(a_{2100}+a_{2010}+a_{1200}+2a_{1110}+a_{1020}+a_{0210}+a_{0120})k + (3k+1)(a_{2001}+2a_{1101}+2a_{1011}+a_{0201}+2a_{0111}+a_{0021})]/(3k+1)^3 \quad (6)$$

According to the inequality constraints in Ref. 3, a_{2100}, a_{2010}, a_{1200}, a_{1110}, a_{1020}, a_{0210} and a_{0120} are all negative, while a_{2001}, a_{0201}, and a_{0021} are all positive. The conditions that the cubic A-patch is single-sheeted are that a_{1002}, a_{0102}, a_{0012}, and a_{0003} must all be positive. This will be the case for $k > 0$ when $a_{1101} + a_{1011} > -a_{2001}$, $a_{1101} + a_{0111} > -a_{0201}$, $a_{1011} + a_{0111} > -a_{0021}$. These three conditions guarantee that a_{1002}, a_{0102}, and a_{0012} are positive, while combined they are equivalent to $a_{1101}+a_{1011}+a_{0111} > -(a_{2001}+a_{0102}+a_{0021})/2$, which guarantees that a_{0003} is positive. Even if these conditions is not satisfied, there may be values of k for which the solution for a_{0003} as given by (6) is positive. These conditions are more easily satisfied the more negative the quantities a_{2100}, a_{2010}, a_{1200}, a_{1110}, a_{1020}, a_{0210} and a_{0120} are.

Next, points on the cubic A-patch are parametrized by lines passing through the singular point and the plane determined by $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. Lines passing through a singular point, or double point, intersect the cubic surface in exactly one more point. These lines have the form

$$\mathbf{L}(t) = t(u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3) + (1-t)\mathbf{p}_0,$$

where $u + v + w = 1$. Thus we make the substitutions

$$\begin{aligned} \alpha_1 &= tu - (1-t)k, & \alpha_2 &= tv - (1-t)k, \\ \alpha_3 &= tw - (1-t)k, & \alpha_4 &= (1-t)(3k+1), \end{aligned} \qquad (7)$$

and (6) into the cubic A-patch (2). This produces an equation which is linear in t: $(P_2 + P_3)t - P_2 = 0$, where

$$\begin{aligned} P_2 \quad &= \quad [k(a_{2100}+a_{2010}) - (3k+1)a_{2001}]u^2 \\ &+ \quad 2[k(a_{2100}+a_{1200}+a_{1110}) - (3k+1)a_{1101}]uv \\ &+ \quad [k(a_{1200}+a_{0210}) - (3k+1)a_{0201}]v^2 \\ &+ \quad 2[k(a_{2010}+a_{1110}+a_{1020}) - (3k+1)a_{1011}]uw \\ &+ \quad [k(a_{1020}+a_{0120}) - (3k+1)a_{0021}]w^2 \\ &+ \quad 2[k(a_{1110}+a_{0210}+a_{0120}) - (3k+1)a_{0111}]vw \end{aligned}$$

and

$$P_3 = a_{2100}u^2v + a_{2010}u^2w + a_{1200}uv^2 + 2a_{1110}uvw + a_{1020}uw^2 + a_{0210}v^2w + a_{0120}vw^2, \qquad (8)$$

so that P_2 and P_3 consist of quadratic and cubic terms in $\{u, v, w\}$, respectively. The region in the uv-plane over which the parametrization takes place can be described by $0 \le u \le 1$, $0 \le 1 - u \le v$. Then t satisfies

$$t = \frac{P_2}{P_2 + P_3}, \quad \text{and} \quad 1 - t = \frac{P_3}{P_2 + P_3}. \qquad (9)$$

Now considering (7), each of α_1, α_2, α_3, and α_4 is seen to be a quotient of cubic polynomials in u, v, and w. Writing $w = 1 - u - v$, each of the α is seen to be a function of two independent variables.

Of particular interest is the situation when $k = 0$, for in that case the cubic splines which are the intersections of the cubic A-patch with the side faces of the tetrahedron are immediately parametrizable. Eqs. (7) with $w = 0$, $v = 0$, and $u = 0$ will parametrize the faces where $\alpha_3 = 0$, $\alpha_2 = 0$, and $\alpha_1 = 0$, respectively. In order for this to work, $\mathbf{p}_4$ must be chosen sufficiently far from $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$. In this case, we have

$$P_2 = -(a_{2001}u^2 + 2a_{1101}uv + 2a_{1011}uw + a_{0201}v^2 + 2a_{0111}vw + a_{0021}w^2). \quad (10)$$

A sufficient condition that the A-patch is single-sheeted in this case is for the denominator in (9) to always have the same sign, say negative, and this can be guaranteed if the coefficients $\{a_{2001}, a_{1101}, a_{1011}, a_{0201}, a_{0111}, a_{0021}\}$ are all positive while $\{a_{2100}, a_{2010}, a_{1200}, a_{1110}, a_{1020}, a_{0210}, a_{0120}\}$ are all negative.

3.2.3. Parametrizing the base triangle in the non-convex case

If the triangle $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ is non-convex, or is convex but not all of its neighbors are convex with the same sign, we are in the non-convex case, and the cubic A-patch intersects $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ in a cubic curve $C(\alpha_1, \alpha_2, \alpha_3) = \sum_{i+j+k=3}[3!/(i!j!k!)]a_{ijk0}\alpha_1^i\alpha_2^j\alpha_3^k = 0$, where $a_{3000} = a_{0300} = a_{0030} = 0$. Let

$$(d, e, f, g, h, i, j) = (3a_{2100}, 3a_{2010}, 3a_{0210}, 3a_{1200}, 3a_{1020}, 3a_{0120}, 6a_{1110}).$$

Then each of $\{d, e, f, g, h, i\}$ is determined, but we still have one degree of freedom left in the coefficients j. This degree of freedom can sometimes be used to make $C(\alpha_1, \alpha_2, \alpha_3)$ rationally parametrizable.

As triangle $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ lies on the plane $\alpha_4 = 0$, it can be regarded as a function of two variables, say x and y, where $(\alpha_1, \alpha_2, \alpha_3) = (x, y, 1 - x - y)$. An irreducible plane cubic curve $F(x, y) = 0$ is singular if it has a double point, that is, a point (x_0, y_0) where $F(x_0, y_0) = F_x(x_0, y_0) = F_y(x_0, y_0) = 0$. By taking resultants of these polynomials and eliminating x_0 and y_0, we obtain this polynomial whose vanishing guarantees either the existence of a double point on $C(\alpha_1, \alpha_2, \alpha_3)$ or that $C(\alpha_1, \alpha_2, \alpha_3)$ is reducible:

$$\begin{aligned} H(j) = {} & t_1 j^6 - t_2 t_3 j^5 + [t_3^2 + t_4(t_2^2 - 12t_1)]j^4 + t_2(8t_3t_4 - t_2^2 + 36t_1)j^3 \\ & + [8(6t_1 - t_2^2)t_4^2 - 8t_3^2t_4 - 6(12t_1 + 5t_2^2)t_3]j^2 \\ & - 4t_2[4t_3t_4^2 + 9(4t_1 - t_2^2)t_4 - 24t_3^2]j \\ & + 8(2t_4^2 - 9t_3)t_4(t_2^2 - 4t_1) + 16t_3^2t_4^2 - 64t_3^3 - 27(t_2^2 - 4t_1)^2, \end{aligned} \quad (11)$$

where

$$t_1 = defghi\ , \quad t_2 = dfh + egi\ , \quad t_3 = defi + dghi + efgh\ , \quad t_4 = di + ef + gh\ .$$

If this $H(j) = 0$ has real solutions, then $C(\alpha_1, \alpha_2, \alpha_3)$ is singular and can be rationally parametrized. If all the solutions of $H(j) = 0$ are complex, then we use the approximate (within any given approximation error) parametrization method given in Ref. 16.

If $H(j) = 0$ is satisfied, then the following is the (3/3) rational parametrization, which is obtained by intersecting the curve with lines $(1-u)(y-y_0) = u(x-x_0)$ through the double point (x_0, y_0):

$$\begin{aligned} x &= \{[-2(e-h)x_0 - t_5 y_0 - (e-2h)](1-u)^3 \\ &\quad + [-t_5 x_0 - 2t_6 y_0 - (2h+2i-j)]u(1-u)^2 \\ &\quad + [-3(f-i)y_0 + (f-2i)]u^2(1-u) + (f-i)x_0 u^3\}/D \\ y &= \{[-t_6 x_0 - 2(f-i)y_0 - (f-2i)]u^3 \\ &\quad + [-2t_5 x_0 - t_6 y_0 - (2h+2i-j)]u^2(1-u) \\ &\quad + [-3(e-h)x_0 + (e-2h)]u(1-u)^2 + (e-h)y_0(1-u)^3\}/D \end{aligned} \tag{12}$$

where

$$\begin{aligned} D &= (e-h)(1-u)^3 + t_5 u(1-u)^2 + t_6 u^2(1-u) + (f-i)u^3 \\ t_5 &= d - e + 2h + i - j \\ t_6 &= -f + g + h + 2i - j \ . \end{aligned}$$

The existence of the condition (11) also provides a method for finding the "best singular approximation" to a nonsingular cubic curve. Given a set $\{d_0, e_0, f_0, g_0, h_0, i_0, j_0\}$, one seeks the value of j, say j_1, nearest j_0 for which (11) is satisfied for the set $\{d_0, e_0, f_0, g_0, h_0, i_0, j_1\}$. All these curves intersect the lines $x = 0$, $y = 0$, and $x + y = 1$ in the same points, namely $(1,0)$, $(0,1)$, $(0,0)$, $(-h/(e-h), 0)$, $(0, -i/(f-i))$, $(-g/(d-g), d/(d-g))$. As j changes continuously from j_0 to j_1, the topology of the cubic curve within the triangle can change only at the endpoint $j = j_1$, a value of j for which the cubic curve is singular. In particular, the same points of intersection with the sides will be connected by non-crossing arcs for all j strictly between j_0 and j_1. Examples of the use of $H(j)$ are given in Appendix D.

4. Conclusions

We have demonstrated how to construct rationally parametrizable A-spline curves and A-patch surfaces on triangles and tetrahedra, respectively. The A-splines interpolate base points on the triangles and are tangent to the corresponding sides. We can construct C^1-continuous quadratic A-splines that have (2/2) rational parametrizations. C^1-continuous irreducible cubic A-splines that have (3/3) rational parametrizations can also be created.

We have also shown how to construct rational parametrizations for C^1-continuous cubic A-patches. These patches interpolate points of a triangulation together with surface normals at the vertices. In addition to the surface itself, the intersections of the surface with the side faces of the tetrahedron containing the patch are rationally parametrizable cubic curves. If a triangle in the triangulation of the surface is non-convex, the intersection of the surface with that triangle (the base triangle of the tetrahedron) may or may not be rationally parametrizable; if it is, a parametrization is given, otherwise an existing approximation [7] can be used. These

NURBS representations allow one to go back and forth efficiently between implicit and parametric forms of these curves and surfaces, thereby allowing one to exploit the advantages of both representations.

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Appendix A:

Proof of condition for rationally parametrizable C^1 quadratic A-spline

From (5) and (2), we have

$$\begin{bmatrix} \mathbf{X}(t) \\ 1 \end{bmatrix} = \frac{1}{w(t)} \begin{bmatrix} \mathbf{p}_1 & w_1\mathbf{p}_3 & \mathbf{p}_2 \\ 1 & w_1 & 1 \end{bmatrix} \begin{bmatrix} B_0^2(t) \\ B_1^2(t) \\ B_2^2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

where $w(t)$ is the denominator of $\mathbf{X}(t)$. From this, we have

$$\alpha_1 = \frac{1}{w(t)} B_0^2(t), \quad \alpha_2 = \frac{1}{w(t)} B_2^2(t), \quad \alpha_3 = \frac{1}{w(t)} w_1 B_1^2(t). \tag{A.1}$$

Substituting these α's into $F_2(\boldsymbol{\alpha}) = 0$ we obtain $\sum_{i=0}^4 c_i B_i^4(t) \equiv 0$, where

$$c_0 = b_{200} \qquad c_1 = b_{101} w_1 \qquad c_2 = \frac{1}{3} b_{110} + \frac{2}{3} b_{002} w_1^2 \qquad c_3 = b_{011} w_1 \qquad c_4 = b_{020}.$$

Since $B_i^4(t),\ i = 0, \ldots, 4$, are linearly independent, we have $c_i = 0, \ \ i = 0, \ldots, 4$. It then follows that

$$b_{110} + 2 b_{002} w_1^2 = 0 \tag{A.2}$$

and hence (6) holds. Summarizing, a C^1 continuous quadratic A-spline $F_2(\boldsymbol{\alpha}) = 2b_{110}\alpha_1\alpha_2 + b_{002}\alpha_3^2 = 0$ has a (2/2) rational parametrization if and only if (A.2), or equivalently (6), holds, and then that parametrization is given by

$$\alpha_1 = \frac{1}{w(t)}B_0^2(t), \quad \alpha_2 = \frac{1}{w(t)}B_2^2(t), \quad \alpha_3 = \frac{1}{w(t)}\sqrt{-\frac{b_{110}}{2b_{002}}}B_1^2(t),$$

where

$$w(t) = B_0^2(t) + \sqrt{-\frac{b_{110}}{2b_{002}}}B_1^2(t) + B_2^2(t). \sqcup$$

Appendix B:

Proof of condition for rationally parametrizable C^1 cubic A-spline

In this appendix we prove that if the coefficients of a C^1 continuous cubic A-spline satisfy (9), then it has a rational parametrization.

From (8) and (A.1) we have

$$\begin{bmatrix} \mathbf{X}(t) \\ 1 \end{bmatrix} =$$

$$\frac{1}{w(t)}\begin{bmatrix} \mathbf{p}_1 & w_1[\mathbf{p}_1 + \alpha(\mathbf{p}_3 - \mathbf{p}_1)] & w_2[\mathbf{p}_2 + \beta(\mathbf{p}_3 - \mathbf{p}_2)] & \mathbf{p}_2 \\ 1 & w_1 & w_2 & 1 \end{bmatrix}\begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

From this, we have

$$\begin{aligned} \alpha_1 &= \frac{1}{w(t)}[B_0^3(t) + w_1(1-\alpha)B_1^3(t)] \\ \alpha_2 &= \frac{1}{w(t)}[w_2(1-\beta)B_2^3(t) + B_3^3(t)] \\ \alpha_3 &= \frac{1}{w(t)}[w_1\alpha B_1^3(t) + w_2\beta B_2^3(t)]. \end{aligned} \tag{B.1}$$

Substituting these α's into $F_3(\alpha_1, \alpha_2, \alpha_3) = 0$, we have $\sum_{i=0}^{9} c_i B_i^9(t) \equiv 0$. Hence we get the following conditions: $c_i = 0, \quad i = 0, \ldots, 9$ where

$$c_0 = b_{300}, \quad c_1 = 9w_1[b_{300}(1-\alpha) + b_{201}\alpha], \quad c_8 = 9w_2[b_{030}(1-\beta) + b_{021}\beta], \quad c_9 = b_{030},$$

affirming that $b_{300} = b_{201} = b_{021} = b_{030} = 0$, and

$$\begin{aligned} c_2 &= 3(3d\,u^2 - a\,v + a\,y) \qquad\qquad \text{(B.2)} \\ c_3 &= -9(3d - f)u^3 + 27d\,u^2x + 18(a - c + d)uv - 18(a - c)uy \end{aligned}$$

$$- 18a\,vx + 18a\,xy + a \tag{B.3}$$

$$\begin{aligned} c_4 &= -27(a-2c+2d+e-f)u^2v + 27(a-2c+e)u^2y \\ &\quad + 54(a-c+d)uvx - 54(a-c)uxy - 27a\,vx^2 + 27a\,x^2y \\ &\quad + 9(b-2c+d)v^2 - 18(b-c)vy + 9b\,y^2 - 6(a-c)u + 6a\,x \end{aligned} \tag{B.4}$$

$$\begin{aligned} c_5 &= -27(b-2c+d+2e-f)uv^2 + 54(b-c+e)uvy - 27b\,uy^2 \\ &\quad + 27(b-2c+d)v^2x - 54(b-c)vxy + 27b\,xy^2 \\ &\quad + 9(a-2c+e)u^2 - 18(a-c)ux + 9a\,x^2 - 6(b-c)v + 6b\,y \end{aligned} \tag{B.5}$$

$$\begin{aligned} c_6 &= -9(3e-f)v^3 + 27e\,v^2y + 18(b-c+e)uv - 18b\,uy \\ &\quad - 18(b-c)vx + 18b\,xy + b \end{aligned} \tag{B.6}$$

$$c_7 = 3(3e\,v^2 - b\,u + b\,x) \tag{B.7}$$

where

$$x = w_1, \quad y = w_2, \quad u = w_1\alpha, \quad v = w_2\beta \tag{B.8}$$

are unknowns and a, b, c, d, e, f are parameters. From (B.2) and (B.7), we have

$$x = u - (3e/b)v^2, \quad y = v - (3d/a)u^2. \tag{B.9}$$

The sign constraints

$$a > 0, \;\; b > 0, \;\; d < 0, \;\; e < 0, \;\; f < 0$$

must be satisfied [4], so it is permissible to divide by a and b here.

Substituting x, y from (B.9) into (B.3) and (B.6) produces

$$0 = 81ade\,u^2v^2 + 9(abf - 6bcd)u^3 + 18abd\,uv + a^2b \tag{B.10}$$

and

$$0 = 81bde\,u^2v^2 + 9(abf - 6ace)v^3 + 18abe\,uv + ab^2 \tag{B.11}$$

for the unknowns u, v. Now when the sum of (B.4) and $3aev^2$ times (B.10) is divided by b, and the substitutions (B.9) are made, we obtain

$$0 = 27a^2ef\,u^3v^2 + 27bd(bd-ae)u^4 + 9ab(af-2cd)u^2v + 3a^2(bd-ae)v^2 + 2a^2bc\,u\,. \tag{B.12}$$

Similarly, when the sum of (B.5) and $3bdu^2$ times (B.11) is divided by a, and the substitutions (B.9) are made, we obtain

$$0 = 27b^2df\,u^2v^3 + 27ae(ae-bd)v^4 + 9ab(bf-2ce)uv^2 + 3b^2(ae-bd)u^2 + 2ab^2c\,v\,. \tag{B.13}$$

We will now show that the system (B.10, B.11, B.12, B.13) has a solution (u, v) when one polynomial constraint is imposed on $\{a, b, c, d, e, f\}$. This will be accomplished by taking resultants of this system by eliminating u and v, and finding that the end results have a factor in common. See Refs. (17, 18) for good discussions on resultants. The resultant of two polynomials vanishes at all values of the parameters where the original polynomials have a simultaneous solution. Let P_1, P_2, P_3, P_4

denote the right-hand sides of (B.10), (B.11), (B.12), (B.13), respectively, and let $R(P_i, P_j; v)$ denote the resultant of P_i and P_j obtained by eliminating the variable v. Then we can say

$$P_5 = R(P_1, P_2; v)/(9a^2b^2) \qquad P_6 = R(P_1, P_3; v)/(729ab^4e)$$
$$P_7 = R(P_1, P_4; v)/(81ab^3) \ . \tag{B.14}$$

Now when the resultant of any two of (P_5, P_6, P_7) obtained by eliminating u is taken, this factor appears:

$$\begin{aligned} G_{[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]}(a,b,c,d,e,f) &= 48a^3e^3f^2 - 9a^2b^2f^4 + 72a^2bcef^3 - 72a^2bde^2f^2 \\ &- 96a^2c^2e^2f^2 - 288a^2cde^3f + 432a^2d^2e^4 + 72ab^2cdf^3 - 72ab^2d^2ef^2 - 8abc^3f^3 \\ &- 552abc^2def^2 + 1152abcd^2e^2f - 864abd^3e^3 + 48ac^4ef^2 + 576ac^3de^2f \\ &- 864ac^2d^2e^3 + 48b^3d^3f^2 - 96b^2c^2d^2f^2 - 288b^2cd^3ef + 432b^2d^4e^2 \\ &+ 48bc^4df^2 + 576bc^3d^2ef - 864bc^2d^3e^2 - 288c^5def + 432c^4d^2e^2. \end{aligned}$$

Consequently, whenever the coefficients (a, b, c, d, e, f) satisfy (9), the polynomials P_5, P_6, and P_7 have a common root for u, and (B.10 – B.13) have a common solution (u, v). ⊔

Appendix C:

Control points computation details

In this appendix we explain the computation of the coefficients of A-patches. The formulas given here are more concrete because all the formulas related to one triangle are provided.

C.1. Convex case

In the convex case, that is, when two adjacent triangles are both positive convex or negative convex, we just need to form four face tetrahedra on the same side of the triangles. This is illustrated in Fig. C.1.

We are given a triangulation of a surface with normal vectors at the vertices, and wish to construct a smooth single-sheeted surface passing through the vertices of the triangulation. Typically, an edge in the triangulation is common to two triangles. Thus for a typical triangle $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$, we let $\mathbf{p}_1'$, $\mathbf{p}_2'$, and $\mathbf{p}_3'$ denote the vertices of the triangulation such that $[\mathbf{p}_1'\mathbf{p}_2\mathbf{p}_3]$, $[\mathbf{p}_1\mathbf{p}_2'\mathbf{p}_3]$, and $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3']$ are also part of the triangulation. Over each of these four triangles we will construct points to make control or "face" tetrahedra; these will be tetrahedra $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$, $[\mathbf{p}_1'\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4']$, $[\mathbf{p}_1\mathbf{p}_2'\mathbf{p}_3\mathbf{p}_4'']$, and $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3'\mathbf{p}_4''']$ (see Fig. C.1).

We also need to introduce "edge" tetrahedra as in [6,8] between the face tetrahedra. Let $\mathbf{p}_1''$, $\mathbf{p}_2''$, and $\mathbf{p}_3''$ be the midpoints of the line segments $[\mathbf{p}_4\mathbf{p}_4']$, $[\mathbf{p}_4\mathbf{p}_4'']$, and $[\mathbf{p}_4\mathbf{p}_4''']$, respectively. Each of these midpoints contributes to the formation of two edge tetrahedra, and these are illustrated in Fig. C.2.

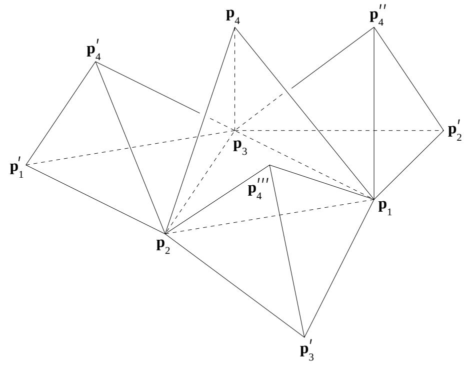


Fig. C.1: Face tetrahedra for a smooth surface

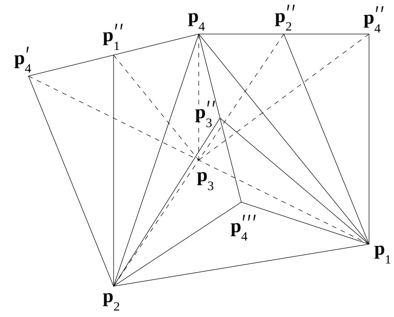


Fig. C.2: Edge tetrahedra for a smooth surface

The following notation will be used to represent the coefficients of all of these tetrahedra:

Face tetrahedron	Weights	Coordinates	Edge tetrahedron	Weights	Coordinates
$\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4$	a^1_{ijkl}	$\boldsymbol{\alpha}^1$	$\mathbf{p}''_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4$	b^1_{ijkl}	$\boldsymbol{\beta}^1$
$\mathbf{p}'_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}'_4$	a^2_{ijkl}	$\boldsymbol{\alpha}^2$	$\mathbf{p}''_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}'_4$	b^2_{ijkl}	$\boldsymbol{\beta}^2$
$\mathbf{p}_1\mathbf{p}'_2\mathbf{p}_3\mathbf{p}''_4$	a^3_{ijkl}	$\boldsymbol{\alpha}^3$	$\mathbf{p}_1\mathbf{p}''_2\mathbf{p}_3\mathbf{p}_4$	b^3_{ijkl}	$\boldsymbol{\beta}^3$
$\mathbf{p}_1\mathbf{p}_2\mathbf{p}'_3\mathbf{p}'''_4$	a^4_{ijkl}	$\boldsymbol{\alpha}^4$	$\mathbf{p}_1\mathbf{p}''_2\mathbf{p}_3\mathbf{p}''_4$	b^4_{ijkl}	$\boldsymbol{\beta}^4$
			$\mathbf{p}_1\mathbf{p}_2\mathbf{p}''_3\mathbf{p}_4$	b^5_{ijkl}	$\boldsymbol{\beta}^5$
			$\mathbf{p}_1\mathbf{p}_2\mathbf{p}''_3\mathbf{p}'''_4$	b^6_{ijkl}	$\boldsymbol{\beta}^6$

We now seek to determine the coefficients of all the face and edge tetrahedra described above. The weights a^1_{ijkl} and b^1_{ijkl}, along with their signs for the case of adjacent convex faces, are depicted in Fig. C.3.

The condition that the cubic A-patch passes through $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}'_1$, $\mathbf{p}'_2$, and $\mathbf{p}'_3$ immediately gives us

$$a^m_{3000} = 0, \quad a^m_{0300} = 0, \quad a^m_{0030} = 0\ , \qquad\qquad m = 1,2,3,4. \tag{C.1}$$

Coefficients around the vertices are given by the normal condition: [9,5]

$$a_{(n-1)e_j+e_k} = a_{ne_j} + \frac{1}{n}(\mathbf{p}_k - \mathbf{p}_j)^T \mathbf{N}_j \ \ , \quad j = 1,2,3,\ k \neq j, \tag{C.2}$$

where the vector

$$\mathbf{N}_j = \nabla S(\mathbf{p}_j) = \left[\frac{\partial S(\mathbf{p}_j)}{\partial x}\ \frac{\partial S(\mathbf{p}_j)}{\partial y}\ \frac{\partial S(\mathbf{p}_j)}{\partial z} \right]^T \quad .$$

C^0 conditions are given by the following [8], [6]

$$\begin{aligned}
&a^1_{0\lambda_2\lambda_3\lambda_4} = b^1_{0\lambda_2\lambda_3\lambda_4}, \quad && a^1_{\lambda_1 0\lambda_3\lambda_4} = b^3_{\lambda_1 0\lambda_3\lambda_4}, \quad && a^1_{\lambda_1\lambda_2 0\lambda_4} = b^5_{\lambda_1\lambda_2 0\lambda_4}, \\
&a^2_{0\lambda_2\lambda_3\lambda_4} = b^2_{0\lambda_2\lambda_3\lambda_4}, \quad && a^3_{\lambda_1 0\lambda_3\lambda_4} = b^4_{\lambda_1 0\lambda_3\lambda_4}, \quad && a^4_{\lambda_1\lambda_2 0\lambda_4} = b^6_{\lambda_1\lambda_2 0\lambda_4}, \\
&b^1_{\lambda_1\lambda_2\lambda_3 0} = b^2_{\lambda_1\lambda_2\lambda_3 0}, \quad && b^3_{\lambda_1\lambda_2\lambda_3 0} = b^4_{\lambda_1\lambda_2\lambda_3 0}, \quad && b^5_{\lambda_1\lambda_2\lambda_3 0} = b^6_{\lambda_1\lambda_2\lambda_3 0},
\end{aligned} \tag{C.3}$$

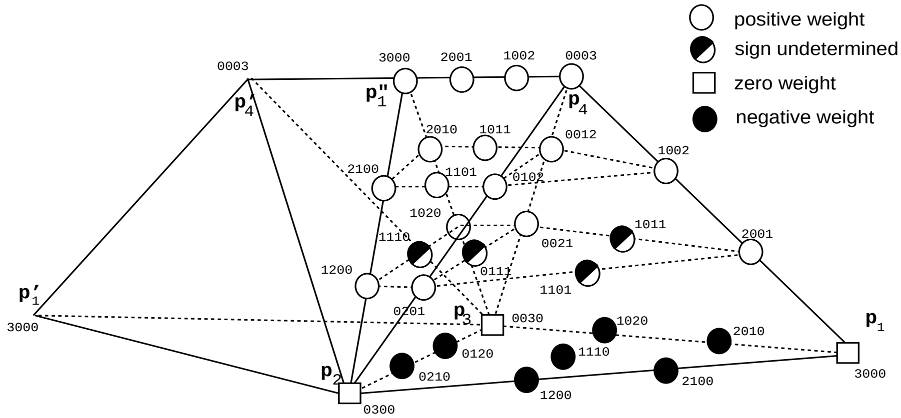


Fig. C.3: Tetrahedra and control points for adjacent convex faces

C^1 conditions between the polynomials on face tetrahedra and edge tetrahedra ([6]):

$$\begin{aligned}
b^1_{1\lambda_2\lambda_3\lambda_4} &= \beta^1_1 a^1_{1\lambda_2\lambda_3\lambda_4} + \beta^1_2 a^1_{0(\lambda_2+1)\lambda_3\lambda_4} + \beta^1_3 a^1_{0\lambda_2(\lambda_3+1)\lambda_4} + \beta^1_4 a^1_{0\lambda_2\lambda_3(\lambda_4+1)} \\
b^2_{1\lambda_2\lambda_3\lambda_4} &= \beta^2_1 a^1_{1\lambda_2\lambda_3\lambda_4} + \beta^2_2 a^1_{0(\lambda_2+1)\lambda_3\lambda_4} + \beta^2_3 a^1_{0\lambda_2(\lambda_3+1)\lambda_4} + \beta^2_4 a^1_{0\lambda_2\lambda_3(\lambda_4+1)} \\
b^3_{\lambda_1 1\lambda_3\lambda_4} &= \beta^3_1 a^1_{(\lambda_1+1)0\lambda_3\lambda_4} + \beta^3_2 a^1_{\lambda_1 1\lambda_3\lambda_4} + \beta^3_3 a^1_{\lambda_1 0(\lambda_3+1)\lambda_4} + \beta^3_4 a^1_{\lambda_1 0\lambda_3(\lambda_4+1)} \\
b^4_{\lambda_1 1\lambda_3\lambda_4} &= \beta^4_1 a^1_{(\lambda_1+1)0\lambda_3\lambda_4} + \beta^4_2 a^1_{\lambda_1 1\lambda_3\lambda_4} + \beta^4_3 a^1_{\lambda_1 0(\lambda_3+1)\lambda_4} + \beta^4_4 a^1_{\lambda_1 0\lambda_3(\lambda_4+1)} \\
b^5_{\lambda_1\lambda_2 1\lambda_4} &= \beta^5_1 a^1_{(\lambda_1+1)\lambda_2 0\lambda_4} + \beta^5_2 a^1_{\lambda_1(\lambda_2+1)0\lambda_4} + \beta^5_3 a^1_{\lambda_1\lambda_2 1\lambda_4} + \beta^5_4 a^1_{\lambda_1\lambda_2 0(\lambda_4+1)} \\
b^6_{\lambda_1\lambda_2 1\lambda_4} &= \beta^6_1 a^1_{(\lambda_1+1)\lambda_2 0\lambda_4} + \beta^6_2 a^1_{\lambda_1(\lambda_2+1)0\lambda_4} + \beta^6_3 a^1_{\lambda_1\lambda_2 1\lambda_4} + \beta^6_4 a^1_{\lambda_1\lambda_2 0(\lambda_4+1)}
\end{aligned} \tag{C.4}$$

where the β^j_i are defined by

$$\begin{aligned}
\mathbf{p}''_1 &= \beta^1_1\mathbf{p}_1 + \beta^1_2\mathbf{p}_2 + \beta^1_3\mathbf{p}_3 + \beta^1_4\mathbf{p}_4 = \beta^2_1\mathbf{p}'_1 + \beta^2_2\mathbf{p}_2 + \beta^2_3\mathbf{p}_3 + \beta^2_4\mathbf{p}'_4 \\
\mathbf{p}''_2 &= \beta^3_1\mathbf{p}_1 + \beta^3_2\mathbf{p}_2 + \beta^3_3\mathbf{p}_3 + \beta^3_4\mathbf{p}_4 = \beta^4_1\mathbf{p}_1 + \beta^4_2\mathbf{p}'_2 + \beta^4_3\mathbf{p}_3 + \beta^4_4\mathbf{p}''_4 \\
\mathbf{p}''_3 &= \beta^5_1\mathbf{p}_1 + \beta^5_2\mathbf{p}_2 + \beta^5_3\mathbf{p}_3 + \beta^5_4\mathbf{p}_4 = \beta^6_1\mathbf{p}_1 + \beta^6_2\mathbf{p}_2 + \beta^6_3\mathbf{p}'_3 + \beta^6_4\mathbf{p}'''_4 \ .
\end{aligned}$$

C^1 conditions between the polynomials on edge tetrahedra [9], [6]:

$$\begin{aligned}
b^1_{(\lambda_1+1)\lambda_2\lambda_3 0} &= \mu^1_1 b^2_{\lambda_1\lambda_2\lambda_3 1} + \mu^1_2 b^1_{\lambda_1(\lambda_2+1)\lambda_3 0} + \mu^1_3 b^1_{\lambda_1\lambda_2(\lambda_3+1)0} + \mu^1_4 b^1_{\lambda_1\lambda_2\lambda_3 1} \\
b^3_{(\lambda_1+1)\lambda_2\lambda_3 0} &= \mu^2_1 b^2_{\lambda_2\lambda_1\lambda_3 1} + \mu^2_2 b^1_{(\lambda_2+1)\lambda_1\lambda_3 0} + \mu^2_3 b^1_{\lambda_2\lambda_1(\lambda_3+1)0} + \mu^2_4 b^1_{\lambda_2\lambda_1\lambda_3 1} \\
b^5_{(\lambda_1+1)\lambda_2\lambda_3 0} &= \mu^3_1 b^2_{\lambda_3\lambda_2\lambda_1 1} + \mu^3_2 b^1_{\lambda_3(\lambda_2+1)\lambda_1 0} + \mu^3_3 b^1_{(\lambda_3+1)\lambda_2\lambda_1 0} + \mu^3_4 b^1_{\lambda_3\lambda_2\lambda_1 1}
\end{aligned} \tag{C.5}$$

where these μ^j_i are defined by

$$\mathbf{p}''_1 = \mu^1_1\mathbf{p}'_4 + \mu^1_2\mathbf{p}_2 + \mu^1_3\mathbf{p}_3 + \mu^1_4\mathbf{p}_4, \quad \mathbf{p}''_2 = \mu^2_1\mathbf{p}_1 + \mu^2_2\mathbf{p}''_4 + \mu^2_3\mathbf{p}_3 + \mu^2_4\mathbf{p}_4,$$
$$\mathbf{p}''_3 = \mu^3_1\mathbf{p}_1 + \mu^3_2\mathbf{p}_2 + \mu^3_3\mathbf{p}'''_4 + \mu^3_4\mathbf{p}_4 \qquad .$$

More equations that we will use come from [6]:

$$\begin{aligned}
a^1_{0111} &= \theta^1_1 a^1_{1110} + \theta^1_2 a^1_{0210} + \theta^1_3 a^1_{0120} + \theta^1_4 a^2_{1110},\\
a^2_{0111} &= \theta^2_1 a^1_{1110} + \theta^2_2 a^2_{0210} + \theta^2_3 a^2_{0120} + \theta^2_4 a^2_{1110},\\
a^1_{1011} &= \theta^3_1 a^1_{2010} + \theta^3_2 a^1_{1110} + \theta^3_3 a^1_{1020} + \theta^3_4 a^3_{1110},\\
a^3_{1011} &= \theta^4_1 a^3_{2010} + \theta^4_2 a^1_{1110} + \theta^4_3 a^3_{1020} + \theta^4_4 a^3_{1110},\\
a^1_{1101} &= \theta^5_1 a^1_{2100} + \theta^5_2 a^1_{1200} + \theta^5_3 a^1_{1110} + \theta^5_4 a^4_{1110},\\
a^4_{1101} &= \theta^6_1 a^4_{2100} + \theta^6_2 a^4_{1200} + \theta^6_3 a^1_{1110} + \theta^6_4 a^4_{1110},
\end{aligned} \tag{C.6}$$

where a^m_{1110} are free and the θ^m_i are given by

$$\begin{array}{ll}
\mathbf{p}_4 = \theta^1_1\mathbf{p}_1 + \theta^1_2\mathbf{p}_2 + \theta^1_3\mathbf{p}_3 + \theta^1_4\mathbf{p}'_1, & \mathbf{p}'_4 = \theta^2_1\mathbf{p}_1 + \theta^2_2\mathbf{p}_2 + \theta^2_3\mathbf{p}_3 + \theta^2_4\mathbf{p}'_1,\\
\mathbf{p}_4 = \theta^3_1\mathbf{p}_1 + \theta^3_2\mathbf{p}_2 + \theta^3_3\mathbf{p}_3 + \theta^3_4\mathbf{p}'_2, & \mathbf{p}''_4 = \theta^4_1\mathbf{p}_1 + \theta^4_2\mathbf{p}_2 + \theta^4_3\mathbf{p}_3 + \theta^4_4\mathbf{p}'_2,\\
\mathbf{p}_4 = \theta^5_1\mathbf{p}_1 + \theta^5_2\mathbf{p}_2 + \theta^5_3\mathbf{p}_3 + \theta^5_4\mathbf{p}'_3, & \mathbf{p}'''_4 = \theta^6_1\mathbf{p}_1 + \theta^6_2\mathbf{p}_2 + \theta^6_3\mathbf{p}_3 + \theta^6_4\mathbf{p}'_3.
\end{array}$$

A simple way of arriving at values of a^m_{1110} is given by [6]:

$$a^m_{1110} = \frac{1}{4}(a^m_{2100} + a^m_{2010} + a^m_{1200} + a^m_{1020} + a^m_{0210} + a^m_{0120}) \ . \tag{C.7}$$

However, there is a more general method for determining these and other coefficients that are necessary for the parametrization of the surface of the edge tetrahedra. This involves computing values that are interpolated averages of the normals of the vertices of the corresponding tetrahedron. For a tetrahedron $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$, define

$$\alpha(\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_l) = \frac{[2(\mathbf{p}_l - \mathbf{p}_i) + (\mathbf{p}_l - \mathbf{p}_j)]^T(\mathbf{p}_j - \mathbf{p}_i)}{\|\mathbf{p}_j - \mathbf{p}_i\|^2} = 1 - \alpha(\mathbf{p}_j,\mathbf{p}_i,\mathbf{p}_l)$$

where $i,j,l \in \{1,2,3,4\}$. Then the interpolation formulas are

$$\begin{aligned}
a^m_{1101} &= [a^m_{2001} + a^m_{0201} + \alpha(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_4)a^m_{1200} + \alpha(\mathbf{p}_2,\mathbf{p}_1,\mathbf{p}_4)a^m_{2100}]/2\\
a^m_{1011} &= [a^m_{0021} + a^m_{0201} + \alpha(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_4)a^m_{2010} + \alpha(\mathbf{p}_1,\mathbf{p}_3,\mathbf{p}_4)a^m_{1020}]/2\\
a^m_{0111} &= [a^m_{0201} + a^m_{2001} + \alpha(\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)a^m_{0120} + \alpha(\mathbf{p}_3,\mathbf{p}_2,\mathbf{p}_4)a^m_{0210}]/2\\
a^m_{1110} &= [a^m_{2010} + a^m_{0210} + \alpha(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)a^m_{1200} + \alpha(\mathbf{p}_2,\mathbf{p}_1,\mathbf{p}_3)a^m_{2100}]/2\\
a^m_{1110} &= [a^m_{0120} + a^m_{2100} + \alpha(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2)a^m_{2010} + \alpha(\mathbf{p}_1,\mathbf{p}_3,\mathbf{p}_2)a^m_{1020}]/2\\
a^m_{1110} &= [a^m_{1200} + a^m_{1020} + \alpha(\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_1)a^m_{0120} + \alpha(\mathbf{p}_3,\mathbf{p}_2,\mathbf{p}_1)a^m_{0210}]/2 \ .
\end{aligned} \tag{C.8}$$

This gives us three ways of computing a^m_{1110} depending on what coefficients are known. If all six of a^m_{ijk0} are known, where $\{i,j,k\} = \{0,1,2\}$ in some order, we use the average of the three formulas, resulting in

$$\begin{aligned}
a^m_{1110} = \frac{1}{2}\Bigg\{ & \frac{[a^m_{1200}(\mathbf{p}_3 - \mathbf{p}_1) - a^m_{2100}(\mathbf{p}_3 - \mathbf{p}_2)]\cdot(\mathbf{p}_2 - \mathbf{p}_1)}{\|\mathbf{p}_2 - \mathbf{p}_1\|^2}\\
& + \frac{[a^m_{2010}(\mathbf{p}_2 - \mathbf{p}_3) - a^m_{1020}(\mathbf{p}_2 - \mathbf{p}_1)]\cdot(\mathbf{p}_1 - \mathbf{p}_3)}{\|\mathbf{p}_1 - \mathbf{p}_3\|^2}\\
& + \frac{[a^m_{0120}(\mathbf{p}_1 - \mathbf{p}_2) - a^m_{0210}(\mathbf{p}_1 - \mathbf{p}_3)]\cdot(\mathbf{p}_3 - \mathbf{p}_2)}{\|\mathbf{p}_3 - \mathbf{p}_2\|^2}\Bigg\} \ .
\end{aligned} \tag{C.9}$$

In Section 3.2.1 it was mentioned that the coefficient b^1_{3000} is free. We select a value for this coefficient to make the trimming curve on triangle $[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3]$, the boundary of the two edge tetrahedra, rationally parametrizable. This is done by finding the values for b^1_{3000} that satisfy (9), where $(a, b, c, d, e, f) = (a^1_{0210}, a^1_{012}, b^1_{1110}, b^1_{2100}, b^1_{2010}, b^1_{3000})$. This equation always has at least two real solutions when the inequality constraints for single-sheetedness in [4] are satisfied. These are $a < 0$, $b < 0$, $d > 0$, $e > 0$ (or with all the inequalities symbols reversed). Then the coefficients of f^0 and f^4 in (9) are $432d^2e^2[(ae - bd)^2 + c^4 - 2c^2(ae + bd)]$ and $-9a^2b^2$. As these are of opposite signs, there must be at least one positive and one negative real value of f satisfying (9). Among these two or four real values, we choose the possible value of b^1_{3000} to be the one that comes closest to satisfying a C^2-continuity condition from [8]. This condition is

$$\begin{aligned} b^i_{2001} \; = \; & {\beta^i_1}^2 a^i_{2001} + 2\beta^i_1\beta^i_2 a^i_{1101} + 2\beta^i_1\beta^i_3 a^i_{1011} + 2\beta^i_1\beta^i_4 a^i_{1002} + {\beta^i_2}^2 a^i_{0201} \\ & + 2\beta^i_2\beta^i_3 a^i_{0111} + 2\beta^i_2\beta^i_4 a^i_{0102} + {\beta^i_3}^2 a^i_{0021} + 2\beta^i_3\beta^i_4 a^i_{0012} + {\beta^i_4}^2 a^i_{0003} \qquad \text{(C.10)} \end{aligned}$$

We obtain estimates for b^1_{2001} and b^2_{2001} using (C.10), and then an estimate for b^1_{3000} by taking their average (this is the first equation of (C.5) where $(\mu^1_1, \mu^1_2, \mu^1_3, \mu^1_4) = (1/2, 0, 0, 1/2)$). The real root of $f = b^1_{3000}$ out of the two or four real roots of (9) that is closest to this estimate is the value chosen for b^1_{3000}.

C.2. Non-convex case

This is the situation when some of the triangles of the triangulation are non-convex and is illustrated in Fig. C.4.

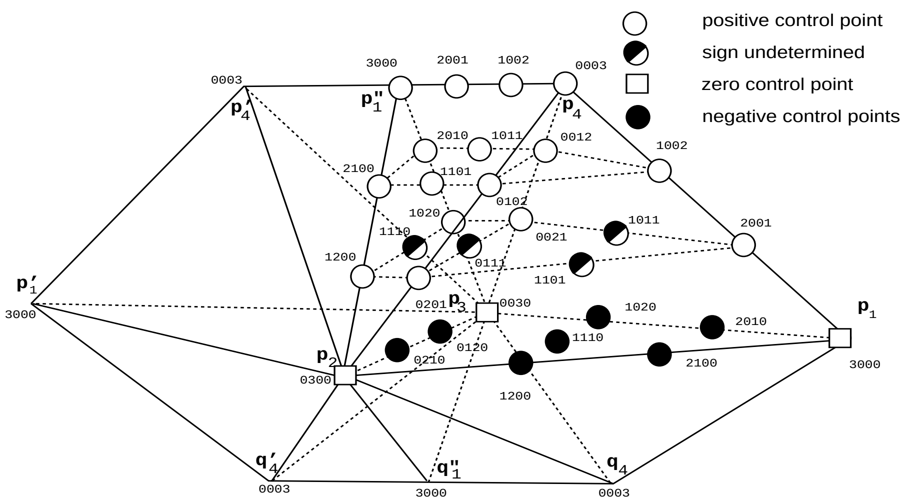


Fig. C.4: Tetrahedra and control points for adjacent non-convex faces

Points $\mathbf{q}_4$, $\mathbf{q}_4'$, $\mathbf{q}_4''$, and $\mathbf{q}_4'''$ are located in positions symmetric to $\mathbf{p}_4$, $\mathbf{p}_4'$, $\mathbf{p}_4''$, and $\mathbf{p}_4'''$ about triangles $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$, $[\mathbf{p}_1'\mathbf{p}_2\mathbf{p}_3]$, $[\mathbf{p}_1\mathbf{p}_2'\mathbf{p}_3]$, and $[\mathbf{p}_1'\mathbf{p}_2\mathbf{p}_3']$, respectively. Points $\mathbf{q}_1''$, $\mathbf{q}_2''$ and $\mathbf{q}_3''$ are the midpoints of $[\mathbf{q}_4\mathbf{q}_4']$, $[\mathbf{q}_4\mathbf{q}_4'']$, and $[\mathbf{q}_4\mathbf{q}_4''']$, respectively. We use the following notation, which is similar to that of Table C.1, for the tetrahedra "below" $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$:

Face tetrahedron	Weights	Coordinates	Edge tetrahedron	Weights	Coordinates
$\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{q}_4$	c^1_{ijkl}	$\boldsymbol{\gamma}^1$	$\mathbf{p}''_1\mathbf{p}_2\mathbf{p}_3\mathbf{q}_4$	d^1_{ijkl}	$\boldsymbol{\delta}^1$
$\mathbf{p}'_1\mathbf{p}_2\mathbf{p}_3\mathbf{q}'_4$	c^2_{ijkl}	$\boldsymbol{\gamma}^2$	$\mathbf{p}''_1\mathbf{p}_2\mathbf{p}_3\mathbf{q}'_4$	d^2_{ijkl}	$\boldsymbol{\delta}^2$
$\mathbf{p}_1\mathbf{p}'_2\mathbf{p}_3\mathbf{q}''_4$	c^3_{ijkl}	$\boldsymbol{\gamma}^3$	$\mathbf{p}_1\mathbf{p}''_2\mathbf{p}_3\mathbf{q}_4$	d^3_{ijkl}	$\boldsymbol{\delta}^3$
$\mathbf{p}_1\mathbf{p}_2\mathbf{p}'_3\mathbf{q}'''_4$	c^4_{ijkl}	$\boldsymbol{\gamma}^4$	$\mathbf{p}_1\mathbf{p}''_2\mathbf{p}_3\mathbf{q}''_4$	d^4_{ijkl}	$\boldsymbol{\delta}^4$
			$\mathbf{p}_1\mathbf{p}_2\mathbf{p}''_3\mathbf{q}_4$	d^5_{ijkl}	$\boldsymbol{\delta}^5$
			$\mathbf{p}_1\mathbf{p}_2\mathbf{p}''_3\mathbf{q}'''_4$	d^6_{ijkl}	$\boldsymbol{\delta}^6$

We then have a set of equations analogous to (C.2 – C.7), with the quantities $(a^m_{ijkl}, b^m_{ijkl}, \mathbf{p}_4, \mathbf{p}'_4, \mathbf{p}''_4, \mathbf{p}'''_4, \mathbf{p}''_1, \mathbf{p}''_2, \mathbf{p}''_3, \beta^j_i, \mu^j_i, \theta^j_i)$ replaced by $(c^m_{ijkl}, d^m_{ijkl}, \mathbf{q}_4, \mathbf{q}'_4, \mathbf{q}''_4, \mathbf{q}'''_4, \mathbf{q}''_1, \mathbf{q}''_2, \mathbf{q}''_3, \tilde{\beta}^j_i, \tilde{\mu}^j_i, \tilde{\theta}^j_i)$.

On some occasions, in order to improve the smoothness properties of the A-patch surface we will partition a non-convex triangle into three smaller triangles using a Clough-Tocher split [6].

Appendix D:

Examples

Here we provide examples of the rational parametrizations for both adjacent convex tetrahedra and adjacent non-convex tetrahedra. We also provide rational parametrizations of the cubic trimming curves in all cases in which it is possible.

D.1. A-patches obtained from surface data

Example 1. This example deals with adjacent convex faces $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ and $[\mathbf{p}'_1\mathbf{p}_2\mathbf{p}_3]$. We obtain the parametrizations of the face tetrahedron $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ and the edge tetrahedron $[\mathbf{p}'_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}'_4]$.

The input data of Example 1

(i,j)	Vertices $\mathbf{p}^{(j)}_i$			Normals $\mathbf{n}^{(j)}_i$			(i,j)	Top vertices $\mathbf{p}^{(j)}_4$		
$(1,0)$	$1,$	$0,$	0	$-2,$	$-3,$	-3	$(4,0)$	$0,$	$0,$	0
$(2,0)$	$0,$	$1,$	0	$-2,$	$-1,$	-2	$(4,1)$	$-1,$	$1/3,$	$2/3$
$(3,0)$	$0,$	$0,$	1	$-2,$	$-3,$	-1	$(4,2)$	$3/4,$	$-3/4,$	$1/4$
$(1,1)$	$-8/3,$	$2,$	$13/3$	$-12,$	$-6,$	-5	$(4,3)$	$2/3,$	$0,$	$-5/6$
$(2,1)$	$9/4,$	$-1,$	$3/4$	$-1,$	$-5,$	-4				
$(3,1)$	$67/42,$	$5/14,$	$-10/21$	$-3,$	$-1,$	-8				

Here $\mathbf{p}''_1 = (\mathbf{p}_4 + \mathbf{p}'_4)/2 = (-1/2, 1/6, 1/3)$, $\mathbf{p}''_2 = (\mathbf{p}_4 + \mathbf{p}''_4)/2 = (3/8, -3/8, 1/8)$, and $\mathbf{p}''_3 = (\mathbf{p}_4 + \mathbf{p}'''_4)/2 = (1/3, 0, -5/12)$, By (C.2) we immediately get

$$
\begin{array}{lllll}
a^1_{2100} = -\frac{1}{3}, & a^1_{2010} = -\frac{1}{3}, & a^1_{2001} = \frac{2}{3}, & a^1_{1200} = -\frac{1}{3}, & a^1_{0210} = -\frac{1}{3}, \\
a^1_{0201} = \frac{1}{3}, & a^1_{1020} = -\frac{1}{3}, & a^1_{0120} = -\frac{2}{3}, & a^1_{0021} = \frac{1}{3}, & a^2_{2100} = -\frac{13}{9}, \\
a^2_{2010} = -\frac{10}{9}, & a^2_{2001} = \frac{25}{9}, & a^2_{1200} = -\frac{13}{9}, & a^2_{0210} = -\frac{1}{3}, & a^2_{0201} = \frac{4}{9}, \\
a^2_{1020} = -\frac{4}{3}, & a^2_{0120} = -\frac{2}{3}, & a^2_{0021} = \frac{4}{9}, & a^3_{2100} = -\frac{7}{12}, & a^3_{2010} = -\frac{1}{3}, \\
a^3_{1200} = -\frac{1}{4}, & a^3_{0210} = -\frac{5}{4}, & a^3_{1020} = -\frac{1}{3}, & a^3_{0120} = -\frac{5}{12}, & a^4_{2100} = -\frac{1}{3}, \\
a^4_{2010} = -\frac{5}{18}, & a^4_{1200} = -\frac{1}{3}, & a^4_{0210} = -\frac{67}{126}, & a^4_{1020} = -\frac{5}{9}, & a^4_{0120} = \frac{1}{9}, \\
b^1_{1200} = \frac{7}{18}, & b^1_{1020} = \frac{7}{18}. & & &
\end{array}
\tag{D.1}
$$

Then by (C.9) we have

$$
a^1_{1110} = -\frac{7}{12} \quad a^2_{1110} = -\frac{46}{25} \quad a^3_{1110} = -\frac{3173}{3675} \quad a^4_{1110} = -\frac{440767}{1474725} \quad .
$$

Next, we find that $(\theta^1_1, \theta^1_2, \theta^1_3, \theta^1_4) = (-1, 3/4, 13/8, -3/8)$, $(\theta^2_1, \theta^2_2, \theta^2_3, \theta^2_4) = (-2, 13/12, 55/24, -3/8)$, $(\theta^3_1, \theta^3_2, \theta^3_3, \theta^3_4) = (9/4, -1, 3/4, -1)$, $(\theta^4_1, \theta^4_2, \theta^4_3, \theta^4_4) = (39/16, -3/2, 13/16, -3/4)$, $(\theta^5_1, \theta^5_2, \theta^5_3, \theta^5_4) = (67/20, 3/4, -1, -21/10)$, and $(\theta^6_1, \theta^6_2, \theta^6_3, \theta^6_4) = (183/40, 7/8, -2, -49/20)$. Therefore from (C.6) we have

$$
\begin{array}{c}
a^1_{0111} = -3/50, \quad a^2_{0111} = -29/900, \\
a^1_{1011} = 2189/4900, \quad a^1_{1101} = -655973/4213500
\end{array}
$$

We now use the idea of putting a singular point on the cubic surface and set $k = 0$ in (6). This gives

$$
a^m_{1002} = a^m_{0102} = a^m_{0012} = a^m_{0003} = 0, \qquad m = 1, 2, 3, 4.
$$

We now find that $(\beta^1_1, \beta^1_2, \beta^1_3, \beta^1_4) = (-1/2, 1/6, 1/3, 1)$ and $(\beta^2_1, \beta^2_2, \beta^2_3, \beta^2_4) = (-3/32, 5/48, 23/96, 3/4)$. Here we again use the idea of putting a singular point on the cubic surface and set $k = 0$ in (6). Now by (C.4) we have

$$
\begin{array}{ll}
b^1_{1110} = -83/1800, & b^1_{1101} = 2866799/25281000, \\
b^1_{1011} = -10783/88200, & b^1_{1002} = 0, \\
b^2_{1110} = -83/1800, & b^2_{1101} = -3/32 a^2_{1101} + 1111/28800, \\
b^2_{1011} = -3/32 a^2_{1011} + 33/320, & b^2_{1002} = 0 \; .
\end{array}
\tag{D.2}
$$

Also, we have $(\mu^1_1, \mu^1_2, \mu^1_3, \mu^1_4) = (1/2, 0, 0, 1/2)$, $(\mu^2_1, \mu^2_2, \mu^2_3, \mu^2_4) = (0, 1/2, 0, 1/2)$, and $(\mu^3_1, \mu^3_2, \mu^3_3, \mu^3_4) = (0, 0, 1/2, 1/2)$. With these values, and the choice $b^1_{2001} = b^2_{2001}$, (C.5) becomes

$$
\begin{array}{ll}
b^1_{3000} = b^1_{2001}, & b^1_{2100} = -3/64 a^2_{1101} + 61472779/808992000, \\
b^1_{2010} = -3/64 a^2_{1011} - 13499/1411200, & b^1_{1110} = -83/1800.
\end{array}
\tag{D.3}
$$

With these values the parametrization (7) for tetrahedron $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ is

$$\begin{aligned}
\alpha_1 &= u(10997235u^2 - 43168002uv + 81208190v^2 \\
&\quad + 23413015u - 81208190v + 34410250)/D\ , \\
\alpha_2 &= v(10997235u^2 - 43168002uv + 81208190v^2 \\
&\quad + 23413015u - 81208190v + 34410250)/D\ , \\
\alpha_3 &= (1-u-v)(10997235u^2 - 43168002uv + 81208190v^2 \\
&\quad + 23413015u - 81208190v + 34410250)/D\ , \\
\alpha_4 &= 17205125(u^2v + 3uv^2 + 2v^3 - 2u^2 - 5uv - 6v^2 + 2u + 4v)/D,
\end{aligned}$$

where

$$\begin{aligned}
D &= 17205125u^2v + 51615375uv^2 + 34410250v^3 - 23413015u^2 \\
&\quad - 129193627uv - 22022560v^2 + 57823265u - 12387690v + 34410250\ .
\end{aligned}$$

The cubic curves which are the intersections of the cubic A-patch with the faces of the tetrahedron have simple rational parametrizations. For the face $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_4]$, where $\alpha_3^1 = 0$ and $w = 1-u-v = 0$, we have

$$\begin{aligned}
\alpha_1^1 &= u(2762723u^2 - 2060473u + 702250)/(1205799u^2 - 854674u + 351125)\ , \\
\alpha_2^1 &= (1-u)(2762723u^2 - 2060473u + 702250)/(1205799u^2 - 854674u + 351125)\ .
\end{aligned}$$

For face $[\mathbf{p}_3\mathbf{p}_1\mathbf{p}_4]$, where $\alpha_2^1 = 0$ and $v = 0$, we have

$$\begin{aligned}
\alpha_1^1 &= u(783u^2 + 1667u + 2450)/(-1667u^2 + 4117u + 2450)\ , \\
\alpha_3^1 &= (1-u)(783u^2 + 1667u + 2450)/(-1667u^2 + 4117u + 2450)\ .
\end{aligned}$$

For face $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$, where $\alpha_1^1 = 0$ and $u = 0$, we have

$$\begin{aligned}
\alpha_2^1 &= v(59v^2 - 59v + 25)/(25v^3 - 16v^2 - 9v + 25)\ , \\
\alpha_3^1 &= (1-v)(59v^2 - 59v + 25)/(25v^3 - 16v^2 - 9v + 25)\ .
\end{aligned}$$

In order to complete the parametrization for the edge tetrahedron $[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ we need specific values for a_{1101}^2, a_{1011}^2, and b_{2001}^1. The first two might be computed in the same way a_{1101}^1 and a_{1011}^1 were, but this requires knowledge of the location of and the surface normals at additional points of the triangulation. Specifically, the triangles involved are those that border triangle $[\mathbf{p}_1'\mathbf{p}_2\mathbf{p}_3]$ on edges $[\mathbf{p}_1'\mathbf{p}_2]$ and $[\mathbf{p}_1'\mathbf{p}_3]$. Alternatively, if triangle $[\mathbf{p}_1'\mathbf{p}_2\mathbf{p}_3]$ is at the boundary of the triangulation, we can use (C.8), which we do in this example. Here we have $\alpha(\mathbf{p}_1', \mathbf{p}_2, \mathbf{p}_4') = 16/11$, $\alpha(\mathbf{p}_3, \mathbf{p}_1', \mathbf{p}_4') = -7/10$, and

$$a_{1101}^2 = 8/9 \quad a_{1011}^2 = 13/15\ .$$

This in turn makes the middle equations of (D.3) become

$$b_{2100}^1 = 27764779/808992000 \quad b_{2010}^1 = -70829/1411200\ .$$

The remaining weight b^1_{3000} is chosen to satisfy (9) in order to make the trimming curve on the face $[\mathbf{p}''_1\mathbf{p}_2\mathbf{p}_3]$ rationally parametrizable. The implicit form of the curve on this face is

$$-{\beta^1_2}^2\beta^1_3 - 2\beta^1_2{\beta^1_3}^2 + \frac{7}{6}\beta^1_1{\beta^1_2}^2 - \frac{83}{300}\beta^1_1\beta^1_2\beta^1_3 + \frac{7}{6}\beta^1_1{\beta^1_3}^2$$
$$+ \frac{27764779}{269664000}{\beta^1_1}^2\beta^1_2 - \frac{70829}{470400}{\beta^1_1}^2\beta^1_3 + b^1_{3000}{\beta^1_1}^3 = 0 \ .$$

Applying the transform (11), with $(\alpha_1, \alpha_2, \alpha_3) = (\beta^1_2, \beta^1_3, \beta^1_1)$, we obtain

$$-{\beta'_2}^2\beta'_3 - 2\beta'_2{\beta'_3}^2 - \frac{611}{275}\beta'_1\beta'_2\beta'_3 - \frac{793362101}{2039334000}{\beta'_1}^2\beta'_2 - \frac{102271}{1185800}{\beta'_1}^2\beta'_3$$
$$+ \left(\frac{64}{1331}b^1_{3000} - \frac{20651204507}{1413258462000}\right){\beta'_1}^3 = 0$$

Substituting the values of the coefficients into (9), we obtain a fourth degree polynomial in b^1_{3000} whose roots are

$$-19.7172, \quad -1.36483, \quad -0.109180, \quad 0.00625911 \ .$$

The estimates for b^1_{2001} and b^2_{2001} according to (C.10) are $b^1_{2001} = 309687311/3716307000$ and $b^2_{2001} = -109/34560$. The estimate for b^1_{3000} is the average of these, which is $1059435803/26427072000 = 0.040089$. The root closest to this estimate is $b^1_{3000} = 0.00625911$. Now we substitute this into (B.14) to obtain the common root $u = 0.796314$ of P_5, P_6, and P_7. Then we substitute into (B.10, B.11, B.12, B.13) to obtain the common root $v = 1.53494$. Next, by (B.9) we get

$$x = 0.491515 \ , \qquad y = 0.794866 \ ,$$

and by (B.8) we have

$$w_1 = 0.491515 \ , \qquad w_2 = 0.794866 \ , \qquad \alpha = 1.62012 \ , \qquad \beta = 1.93106 \ .$$

These values give the parametrization for the cubic trimming curve in triangle $[\mathbf{p}''_1\mathbf{p}_2\mathbf{p}_3]$

$$\begin{aligned}
\beta'_2 &= [B^3_0(t) - 0.30480B^3_1(t)]/D \ , \\
\beta'_3 &= [-0.74007B^3_1(t) + B^3_0(t)]/D \ , \\
\beta'_1 &= [0.79631B^3_1(t) + 1.53493B^3_2(t)]/D \ ,
\end{aligned}$$

where

$$D = B^3_0(t) + 0.49152B^3_1(t) + 0.79487B^3_2(t) + B^3_3(t) \ .$$

Returning to the original coordinates by using the transformation (11), we have

$$\begin{aligned}
\beta^1_2 &= [B^3_0(t) - 0.13588B^3_1(t) + 0.32559B^3_2(t)]/D \ , \\
\beta^1_3 &= [1.01349B^3_1(t) - 0.26665B^3_2(t) + B^3_3(t)]/D \ , \\
\beta^1_1 &= [0.86871B^3_1(t) + 1.67448B^3_2(t)]/D \ ,
\end{aligned}$$

where

$$D = B_0^3(t) + 0.49152B_1^3(t) + 0.79487B_2^3(t) + B_3^3(t) \; .$$

With the above values, including $b_{3000}^1 = b_{2100}^1 = 0.00625911$, the parametrization (7) for tetrahedron $[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ is

$$\begin{aligned}
\beta_1^1 &= u(0.584105u^2 + 1.25797uv + 0.786667v^2 \\
&\quad - 0.911179u - 0.786667v + 0.333333)/D \; , \\
\beta_2^1 &= v(0.584105u^2 + 1.25797uv + 0.786667v^2 \\
&\quad - 0.911179u - 0.786667v + 0.333333)/D \; , \\
\beta_3^1 &= (1-u-v)(0.584105u^2 + 1.25797uv + 0.786667v^2 \\
&\quad - 0.911179u - 0.786667v + 0.333333)/D \; , \\
\beta_4^1 &= (-0.439080u^3 - 0.287844u^2v - 0.130000uv^2 + 0.333333v^3 \\
&\quad + 0.827968u^2 - 0.463333uv - 1.00000v^2 - 0.388889u + 0.666667v)/D,
\end{aligned}$$

where

$$\begin{aligned}
D = &-0.439080u^3 - 0.287844u^2v + 0.130000uv^2 + 0.333333v^3 \\
&+ 1.41207u^2 + 0.794641uv - 0.213333v^2 - 1.30007u - 0.120000v + 0.333333 \; .
\end{aligned}$$

The cubic curves that are the intersections of the cubic A-patch with three of the faces of $[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ have rational parametrizations. For the face $[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_4]$, where $\beta_3^1 = 0$ and $w = 1 - u - v = 0$, we have

$$\begin{aligned}
\beta_1^1 &= u(0.112798u^2 - 0.439872u + 0.333333) \\
&\quad /(-0.354569u^3 + 0.856255u^2 - 0.828761u + 0.333333) \; , \\
\beta_2^1 &= (1-u)(0.112798u^2 - 0.439872u + 0.333333) \\
&\quad /(-0.354569u^3 + 0.856255u^2 - 0.828761u + 0.333333) \; .
\end{aligned}$$

However, this curve lies outside triangle $[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ for all u in $(0, 1)$, so the surface patch does not intersect this triangle except at the vertices $(0, 1, 0)$ and $(1, 0, 0)$. For face $[\mathbf{p}_3\mathbf{p}_1''\mathbf{p}_4]$, where $\beta_2^1 = 0$ and $v = 0$, we have

$$\begin{aligned}
\beta_1^1 &= u(0.584105u^2 - 0.911179u + 0.333333) \\
&\quad /(-0.439080u^3 + 1.41207u^2 - 1.30007u + 0.333333) \; , \\
\beta_3^1 &= (1-u)(0.584105u^2 - 0.911179u + 0.333333) \\
&\quad /(-0.439080u^3 + 1.41207u^2 - 1.30007u + 0.333333) \; , \\
&\qquad 0.585831 \leq u \leq 0.885691 \; .
\end{aligned}$$

The lower limit for the range of u is where $\beta_1^1 = \beta_3^1 = 0$, and the upper limit is where $\beta_1^1 + \beta_3^1 = 1$. For face $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$, where $\beta_1^1 = 0$ and $u = 0$, we have a parametrization that is equivalent for that of tetrahedron $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$:

$$\beta_2^1 = v(0.786667v^2 - 0.786667v + 0.333333)$$

$$
\begin{aligned}
& \quad /(0.333333v^3 - 0.213333v^2 - 0.120000v + 0.333333) \, , \\
\beta_3^1 \;\; &= \;\; (1-v)(0.786667v^2 - 0.786667v + 0.333333) \\
& \quad /(0.333333v^3 - 0.213333v^2 - 0.120000v + 0.333333) \, , \\
& \qquad\qquad\qquad\qquad\qquad\qquad 0 \le v \le 1 \;\; .
\end{aligned}
$$

Parametrizations for the other edge tetrahedra surrounding $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ can be found in a similar manner.

Example 2. In this example, the input data consists of one convex triangle $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ and one non-convex triangle $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1']$. The non-convex triangle is further subdivided into three sub-triangles $[\mathbf{c}\mathbf{p}_3\mathbf{p}_1']$, $[\mathbf{p}_2\mathbf{c}\mathbf{p}_1']$ and $[\mathbf{p}_2\mathbf{p}_3\mathbf{c}]$, where $\mathbf{c}$ is the centroid $(\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_1')/3$. The edge $[\mathbf{p}_2\mathbf{p}_3]$ between the two triangles is convex. The output data are the parameters of the NURBS representation for the face and edge A-patches, as well as the NURBS representation of the boundary curve. Here we only need to give the NURBS representation for the curve on the face between the two edge tetrahedra, since the other trimming curves are easily obtained by restricting the domain parameters of their corresponding tetrahedra to their boundaries.

The input data of Example 2

(i,j)	Vertices $\mathbf{p}_i^{(j)}$			Normals $\mathbf{n}_i^{(j)}$		
$(1,0)$	0.000000,	3.000000,	−0.375000	0.000000,	1.000000,	1.500000
$(2,0)$	−1.500000,	0.000000,	0.375000	−1.000000,	0.000000,	1.000000
$(3,0)$	1.500000,	0.000000,	0.375000	1.000000,	0.000000,	1.000000
$(1,1)$	0.000000,	−3.000000,	−0.375000	0.000000,	1.000000,	1.500000
(i,j)	Top vertices $\mathbf{p}_4^{(j)}$			Bottom vertices $\mathbf{q}_4^{(j)}$		
$(4,0)$	0.000000,	5.099788,	16.524153	0.000000,	−2.662288,	−14.524153
$(4,1)$	0.000000,	−5.099788,	16.524153	0.000000,	3.147359,	−16.464437

The notations $\mathbf{p}_i^{(0)}$ and $\mathbf{p}_i^{(1)}$ refer to $\mathbf{p}_i$ and $\mathbf{p}_i'$.

The output NURBS parameters of Example 2

Elements	Labels	Parameters
$[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1\mathbf{p}_4]$	P_2	4.883, 6.665, 9.149, 6.665, 4.883, 3.910,
$[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1]$	P_3	−0.750, −0.625, −0.625, −0.750, −1.000, −1.000, −1.202
$[\mathbf{c}\mathbf{p}_3\mathbf{p}_1'\mathbf{p}_4']$	P_2	4.883, 6.346, 7.749, 6.813, 5.617, 5.019,
$[\mathbf{c}\mathbf{p}_3\mathbf{p}_1']$	P_3	−0.750, 1.375, 0.916, 0.436, −0.490, −0.583, 0.196
$[\mathbf{c}\mathbf{p}_3\mathbf{p}_1'\mathbf{q}_4']$	P_2	−6.113, −6.024, −5.995, −6.014, −6.171, −6.250,
$[\mathbf{p}_2\mathbf{c}\mathbf{p}_1'\mathbf{p}_4']$	P_2	5.617, 6.813, 7.749, 6.346, 4.883, 5.019,
$[\mathbf{p}_2\mathbf{c}\mathbf{p}_1']$	P_3	0.436, 0.916, 1.375, −0.750, −0.583, −0.490, 0.196
$[\mathbf{p}_2\mathbf{c}\mathbf{p}_1'\mathbf{q}_4']$	P_2	−6.171, −6.014, −5.995, −6.024, −6.113, −6.250,
$[\mathbf{p}_2\mathbf{p}_3\mathbf{c}\mathbf{p}_4']$	P_2	4.883, 5.019, 5.617, 5.019, 4.883, 3.829,
$[\mathbf{p}_2\mathbf{p}_3\mathbf{c}]$	P_3	−0.583, −0.490, −0.490, −0.583, −1.000, −1.000, −1.083
$[\mathbf{p}_2\mathbf{p}_3\mathbf{c}\mathbf{q}_4']$	P_2	−6.113, −6.250, −6.171, −6.250, −6.113, −6.613,
$[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$	P_2	4.883, 3.910, 4.883, −3.251, 1.610, −3.251,
$[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1'']$	P_3	−1.000, −1.000, 4.883, −3.033, 1.610, −3.033, 4.883, 3.870
$[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4']$	P_2	4.883, 3.829, 4.883, −2.815, 1.610, −2.815,
$[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1''']$	C_{01}	0.927, 0.927, 0.822, 0.822

Here $\mathbf{p}_1''' = (0.000, 0.000, 1.875)$, $\mathbf{p}_1'' = (\mathbf{p}_4 + \mathbf{p}_4')/2$, and $C_{01} = (w_1, w_2, \alpha, \beta)$, which define a rational Bézier curve by (5) for a specified triangle. The point $\mathbf{p}_1'''$ is the point of intersection of the tangent lines to this curve at $\mathbf{p}_2$ and $\mathbf{p}_3$, and the triangle $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1''']$ is the sub-triangle of $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1'']$ as illustrated in Fig. 1(c). (The points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ in Fig. 1(c) correspond to the points $\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_1'''$ here.) P_2 and P_3 are given by (10) and (8), respectively. The coefficients of P_2 are arranged in the order of $a_{0201}, a_{0111}, a_{0021}, a_{1011}, a_{2001}, a_{1101}$. The order of P_3 for face tetrahedra is $a_{0210}, a_{0120}, a_{1020}$, $a_{2010}, a_{2100}, a_{1200}, a_{1110}$, and for edge tetrahedra is $a_{0210}, a_{0120}, a_{1020}, a_{2010}, a_{3000}, a_{2100}, a_{1200}$, a_{1110}. Fig. D.1 shows the input triangles, normals, and the piecewise smooth surface patches as well as isophotes on the surface. The continuous isophotes demonstrate that the composite surface is smooth.

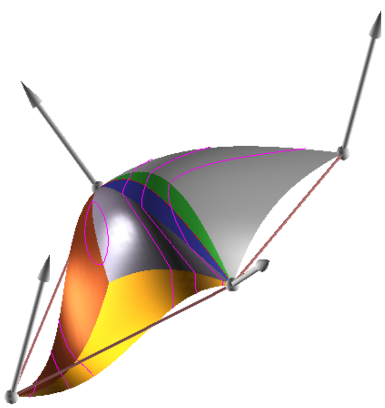


Fig. D.1: A convex face patch and a non-convex face patch joined with convex edge patches

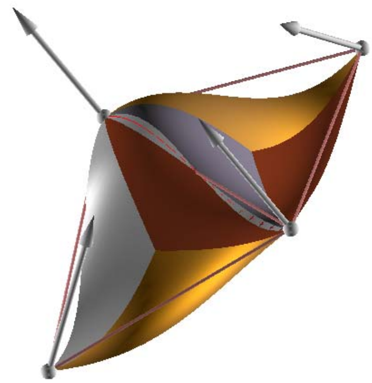


Fig. D.2: Two non-convex face patches joined with non-convex edge patches

We can parametrize the non-convex base triangles $[\mathbf{c}\mathbf{p}_3\mathbf{p}_1']$ and $[\mathbf{p}_2\mathbf{c}\mathbf{p}_1']$ through the use of (11) and (12). For triangle $[\mathbf{c}\mathbf{p}_3\mathbf{p}_1']$, we have $(d, e, f, g, h, i, j) = (-1.470, 1.308, -2.250, -1.749, 2.748, 4.125, 1.176)$. Using these values for (d, e, f, g, h, i), we find that the value of j nearest 1.176 that satisfies (11) is $j_1 = 0.880$. The double point of the singular cubic is then $(x_0, y_0) = (-0.513, 1.063)$, and the parametrization (12) is

$$\begin{aligned} x &= [-0.673(1-u)^3 + 6.649u(1-u)^2 + 9.826u^2(1-u) + 3.269u^3]/D \\ y &= [1.530(1-u)^3 + 6.403u(1-u)^2 + 7.695u^2(1-u) + 2.394u^3]/D \end{aligned}$$

where

$$D = 1.440(1-u)^3 + 5.963u(1-u)^2 + 10.619u^2(1-u) + 6.375u^3 \quad .$$

The portion of this curve within triangles $[\mathbf{c}\mathbf{p}_3\mathbf{p}_1']$ is that for $-4.287 \le u \le -2.362$.

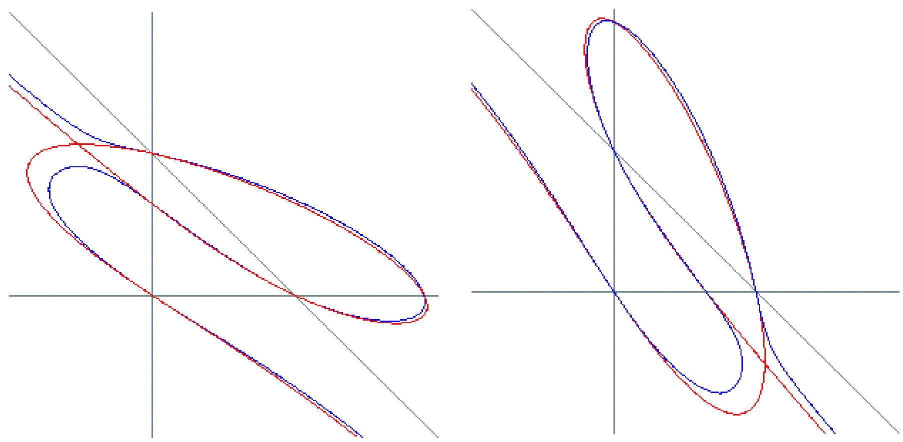
For triangle $[\mathbf{p}_2\mathbf{c}\mathbf{p}_1']$, we have $(d,e,f,g,h,i,j) = (-1.749, -2.250, 1.308, -1.470, 4.125, 2.748, 1.176)$. Using these values for (d,e,f,g,h,i), we find that the value of j nearest 1.176 that satisfies (11) is again $j_1 = 0.880$. This time the double point of the singular cubic is $(x_0, y_0) = (1.063, -0.513)$, and the parametrization (12) is

$$\begin{aligned} x &= [2.394(1-u)^3 + 7.695u(1-u)^2 + 6.403u^2(1-u) + 1.530u^3]/D \\ y &= [-3.268(1-u)^3 - 9.826u(1-u)^2 - 6.649u^2(1-u) - 0.673u^3]/D \end{aligned}$$

where

$$D = 6.375(1-u)^3 + 10.619u(1-u)^2 + 5.963u^2(1-u) + 1.440u^3 \ .$$

The portion of this curve within triangles $[\mathbf{c}\mathbf{p}_3\mathbf{p}_1']$ is that for $-4.287 \le u \le -2.362$. Plots of these two singular curves, along with the actual curves they approximate, are shown in Figures D.3(a) and (b). In both cases the actual and approximating singular cubic curves are virtually indistinguishable within the base triangles.



(a) (b)

Fig. D.3: Trimming curve (blue) and approximating singular curve (red) for (a) triangle $[\mathbf{c}\mathbf{p}_3\mathbf{p}_1']$ and (b) triangle $[\mathbf{p}_2\mathbf{c}\mathbf{p}_1']$. The x- and y-axes and the line $x+y=1$ are shown in gray, where $x = \alpha_1$ and $y = \alpha_2$ in the respective coordinate systems of the triangles.

Example 3. In this example, the input data consists of two non-convex triangles $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ and $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1']$. Each of the triangles is further subdivided into three subtriangles at its center. The edge $[\mathbf{p}_2\mathbf{p}_3]$ between the two triangles is non-convex. The meaning of the output data is the same as that of Example 2. However, since the edge $[\mathbf{p}_2\mathbf{p}_3]$ is non-convex, the boundary curve corresponding to this edge is broken into two (see Fig. 1(b)).

The input data of Example 3

(i,j)	Vertices $\mathbf{p}_i^{(j)}$			Normals $\mathbf{n}_i^{(j)}$		
$(1,0)$	0.000000,	3.000000,	−0.375000	0.000000,	−1.000000,	1.500000
$(2,0)$	−1.500000,	0.000000,	0.375000	1.000000,	0.000000,	1.000000
$(3,0)$	1.500000,	0.000000,	0.375000	1.000000,	0.000000,	1.000000
$(1,1)$	0.000000,	−3.000000,	−0.375000	0.000000,	1.000000,	1.500000
(i,j)	Top vertices $\mathbf{p}_4^{(j)}$			Bottom vertices $\mathbf{q}_4^{(j)}$		
$(4,0)$	0.000000,	5.099788,	16.524153	0.000000,	−2.662288,	−14.524153
$(4,1)$	0.000000,	−5.099788,	16.524153	0.000000,	3.147359,	−16.464437

The output NURBS parameters of Example 3

Elements	Labels	Parameters							
$[\mathbf{c}_0\mathbf{p}_3\mathbf{p}_1\mathbf{p}_4]$	P_2	4.883,	6.346,	7.749,	7.083,	6.253,	5.537,		
$[\mathbf{c}_0\mathbf{p}_3\mathbf{p}_1]$	P_3	−0.750,	1.375,	0.916,	0.706,	−0.156,	−0.583,	0.196	
$[\mathbf{c}_0\mathbf{p}_3\mathbf{p}_1\mathbf{q}_4]$	P_2	−6.113,	−6.024,	−5.995,	−5.745,	−5.658,	−5.916,		
$[\mathbf{p}_2\mathbf{c}_0\mathbf{p}_1\mathbf{p}_4]$	P_2	6.253,	7.083,	7.749,	7.155,	5.883,	6.140,		
$[\mathbf{p}_2\mathbf{c}_0\mathbf{p}_1]$	P_3	0.706,	0.916,	1.375,	0.250,	0.416,	0.446,	1.005	
$[\mathbf{p}_2\mathbf{c}_0\mathbf{p}_1\mathbf{q}_4]$	P_2	−5.658,	−5.745,	−5.995,	−5.215,	−5.113,	−5.313,		
$[\mathbf{p}_2\mathbf{p}_3\mathbf{c}_0\mathbf{p}_4]$	P_2	4.883,	5.537,	6.253,	6.140,	5.883,	5.383,		
$[\mathbf{p}_2\mathbf{p}_3\mathbf{c}_0]$	P_3	−0.583,	−0.156,	0.446,	0.416,	1.000,	−1.000,	−0.083	
$[\mathbf{p}_2\mathbf{p}_3\mathbf{c}_0\mathbf{q}_4]$	P_2	−6.113,	−5.916,	−5.658,	−5.313,	−5.113,	−5.613,		
$[\mathbf{c}_1\mathbf{p}_3\mathbf{p}_1'\mathbf{p}_4']$	P_2	4.883,	6.346,	7.749,	7.083,	6.253,	5.537,		
$[\mathbf{c}_1\mathbf{p}_3\mathbf{p}_1']$	P_3	−0.750,	1.375,	0.916,	0.706,	−0.156,	−0.583,	0.196	
$[\mathbf{c}_1\mathbf{p}_3\mathbf{p}_1'\mathbf{q}_4']$	P_2	−6.113,	−6.024,	−5.995,	−5.745,	−5.658,	−5.916,		
$[\mathbf{p}_2\mathbf{c}_1\mathbf{p}_1'\mathbf{p}_4']$	P_2	6.253,	7.083,	7.749,	7.155,	5.883,	6.140,		
$[\mathbf{p}_2\mathbf{c}_1\mathbf{p}_1']$	P_3	0.706,	0.916,	1.375,	0.250,	0.416,	0.446,	1.005	
$[\mathbf{p}_2\mathbf{c}_1\mathbf{p}_1'\mathbf{q}_4']$	P_2	−5.658,	−5.745,	−5.995,	−5.215,	−5.113,	−5.313,		
$[\mathbf{p}_2\mathbf{p}_3\mathbf{c}_1\mathbf{p}_4']$	P_2	4.883,	5.537,	6.253,	6.140,	5.883,	5.383,		
$[\mathbf{p}_2\mathbf{p}_3\mathbf{c}_1]$	P_3	−0.583,	−0.156,	0.446,	0.416,	1.000,	−1.000,	−0.083	
$[\mathbf{p}_2\mathbf{p}_3\mathbf{c}_1\mathbf{q}_4']$	P_2	−6.113,	−5.916,	−5.658,	−5.313,	−5.113,	−5.613,		
$[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$	P_2	5.883,	5.383,	4.883,	−1.535,	0.423,	−1.986,		
$[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1'']$	P_3	1.000,	−1.000,	4.883,	−1.535,	0.423,	−1.986,	5.883,	5.383
$[\mathbf{p}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4']$	P_2	5.883,	5.383,	4.883,	−1.535,	0.423,	−1.986,		
$[\mathbf{q}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{q}_4]$	P_2	−5.113,	−5.613,	−6.113,	−0.423,	−0.015,	−0.091,		
$[\mathbf{p}_2\mathbf{p}_3\mathbf{q}_1'']$	P_3	1.000,	−1.000,	−6.113,	−0.423,	−0.015,	−0.091,	−5.113,	−5.613
$[\mathbf{q}_1''\mathbf{p}_2\mathbf{p}_3\mathbf{q}_4']$	P_2	−5.113,	−5.613,	−6.113,	−0.423,	−0.015,	−0.091,		
$[\mathbf{c}_2\mathbf{p}_3\mathbf{p}_1''']$	C_{01}	0.972,	0.986,	0.534,	1.039				
$[\mathbf{p}_2\mathbf{c}_2\mathbf{q}_1''']$	C_{01}	1.010,	0.995,	1.030,	0.517				

Here $\mathbf{p}_1''' = (1.000, 0.000, 1.875)$, $\mathbf{q}_1''' = (-1.000, 0.000, -0.125)$, $\mathbf{c}_0 = (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)/3$, $\mathbf{c}_1 = (\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_1')/3$, $\mathbf{c}_2 = (\mathbf{p}_2 + \mathbf{p}_3)/2$, $\mathbf{p}_1'' = (\mathbf{p}_4 + \mathbf{p}_4')/2$, and $\mathbf{q}_1'' = (\mathbf{q}_4 + \mathbf{q}_4')/2$. The point $\mathbf{q}_1'''$ is analogous to $\mathbf{p}_1'''$, with the former being a vertex of the subtriangle of $[\mathbf{p}_2\mathbf{p}_3\mathbf{q}_1'']$ formed by tangent lines at $\mathbf{p}_2$ and $\mathbf{p}_3$. Note that the curves on $[\mathbf{c}_2\mathbf{p}_3\mathbf{p}_1''']$ and $[\mathbf{p}_2\mathbf{c}_2\mathbf{q}_1''']$ are on the faces $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_1'']$ and $[\mathbf{p}_2\mathbf{p}_3\mathbf{q}_1'']$, respectively. Fig. D.2 shows the input triangles, normals, and the piecewise smooth surface.

Example 4: Here we obtain the rational parametrization of a cubic A-patch given its implicit representation and two of its trimming curves in rational parametric form. We use the surface and trimming curves on the faces of triangles $[\mathbf{p}_3\mathbf{p}_1\mathbf{p}_4]$ and $[\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$ from Example 1. These are

$$\begin{aligned} f(\alpha_1,\alpha_2,\alpha_3) = {} & -2\alpha_1^3 - 1450777/702250\alpha_1^2\alpha_2 - 13917/2450\alpha_1^2\alpha_3 \\ & - 748527/702250\alpha_1\alpha_2^2 - 168139023/34410250\alpha_1\alpha_2\alpha_3 - 11467/2450\alpha_1\alpha_3^2 - \alpha_2^3 \\ & - 41/25\alpha_2^2\alpha_3 - 66/25\alpha_2\alpha_3^2 - \alpha_3^3 + 2\alpha_1^2 - 655973/702250\alpha_1\alpha_2 \\ & + 6567/2450\alpha_1\alpha_3 + \alpha_2^2 - 9/25\alpha_2\alpha_3 + \alpha_3^2 = 0 \end{aligned}$$

and

$$\begin{aligned} \mathbf{c}_1 &= \begin{bmatrix} \dfrac{u(783u^2+1667u+2450)}{-1667u^2+4117u+2450} & 0 & \dfrac{(1-u)(783u^2+1667u+2450)}{-1667u^2+4117u+2450} \end{bmatrix}^T \\ \mathbf{c}_2 &= \begin{bmatrix} 0 & \dfrac{v(59v^2-59v+25)}{25v^3-16v^2-9v+25} & \dfrac{(1-v)(59v^2-59v+25)}{25v^3-16v^2-9v+25} \end{bmatrix}^T \quad . \end{aligned}$$

The parametrization formula (5) gives

$$\begin{aligned} \alpha_1 = {} & u(783u^2+1667u+2450)(59v^2-59v+25) \\ & [783(-164980287v^4 + 477749781v^3 - 492419089v^2 \\ & + 214059845v - 34410250)u^3 \\ & + (-617395867469v^4 + 1535042794507v^3 - 1641986158653v^2 \\ & + 754757892615v - 141666999250)u^2 \\ & + 490(685902361v^4 - 398728537v^3 - 490404974v^2 + 258217325v)u \\ & + 3372204500(118v^4 - 186v^3 + 43v^2)]/D \\ \alpha_2 = {} & v(783u^2+1667u+2450)(59v^2-59v+25) \\ & [10997235(2108v^3 - 6324v^2 + 5441v - 2450)u^4 \\ & + 2(-188770087307v^3 + 289020881567v^2 \\ & - 172470157210v + 26943225750)u^3 \\ & + (-424647388056v^3 + 944813401296v^2 \\ & - 660730182965v + 141666999250)u^2 \\ & + 1376410(93119v^3 + 73581v^2 - 91925v)u + 3372204500(93v^3 - 43v^2)]/D \\ \alpha_3 = {} & 49(783u^2+1667u+2450)(59v^2-59v+25) \\ & [783(2762723v^4 - 7333109v^3 + 6677136v^2 - 2106750v)u^4 \\ & + 2(8834249769v^4 - 21597414834v^3 + 22238817240v^2 \\ & - 10060977550v + 1720512500)u^3 \\ & + (-10792666047v^4 + 7366509965v^3 + 4160323732v^2 - 3554403650v)u^2 \\ & + 50(-77550162v^4 + 187243001v^3 - 64285354v^2)u \\ & + 1720512500(v^4 - 2v^3)]/D \end{aligned}$$

where

$$
\begin{aligned}
D = {} & 8610835005(52700v^6 - 191828v^5 + 218237v^4 - 38690v^3 \\
& - 167869v^2 + 158075v - 61250)u^6 \\
& + 783(8021667692461v^6 - 47214015347250v^5 + 92929843987272v^4 \\
& \quad - 98075068184563v^3 + 57722349056705v^2 \\
& \quad - 17712768936500v + 1347161287500)u^5 \\
& + (6718080542543844v^6 - 36650126689750370v^5 \\
& \quad + 100358520286450013v^4 - 144753135630488742v^3 \\
& + 114335297781470505v^2 - 46449093712737125v + 6499417482818750)u^4 \\
& + (-24000847573657227 7v^6 + 749134008140527950v^5 \\
& - 1049038201045620894v^4 + 791095191775983271v^3 \\
& - 326743055580646175v^2 + 66528582744601250v - 7026831126875000)u^3 \\
& + 490(-125046071634397v^6 + 717177371446119v^5 - 1314985300463672v^4 \\
& \quad + 1126445756409325v^3 - 452026933696750v^2 + 62986024675000v)u^2 \\
& + 6002500(27681537447v^6 - 56487253058v^5 + 37858019080v^4 \\
& \quad - 5480976369v^3 - 2521112225v^2)u \\
& + 8261901025000(9287v^6 - 20499v^5 + 16312v^4 - 4475v^3).
\end{aligned}
$$

In order to parametrize the cubic A-patch within tetrahedron $[\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4]$, we need ranges of u and v where α_1, α_2, α_3, and α_4 are between 0 and 1, or equivalently, where $D\alpha_1$, $D\alpha_2$, $D\alpha_3$, and $D\alpha_4$ all have the same sign. Seven of these regions in the uv-plane where this is the case are shown in Fig. D.4. The x and y-axes are parts of the graphs of $\alpha_2 = 0$ and $\alpha_1 = 0$, respectively.

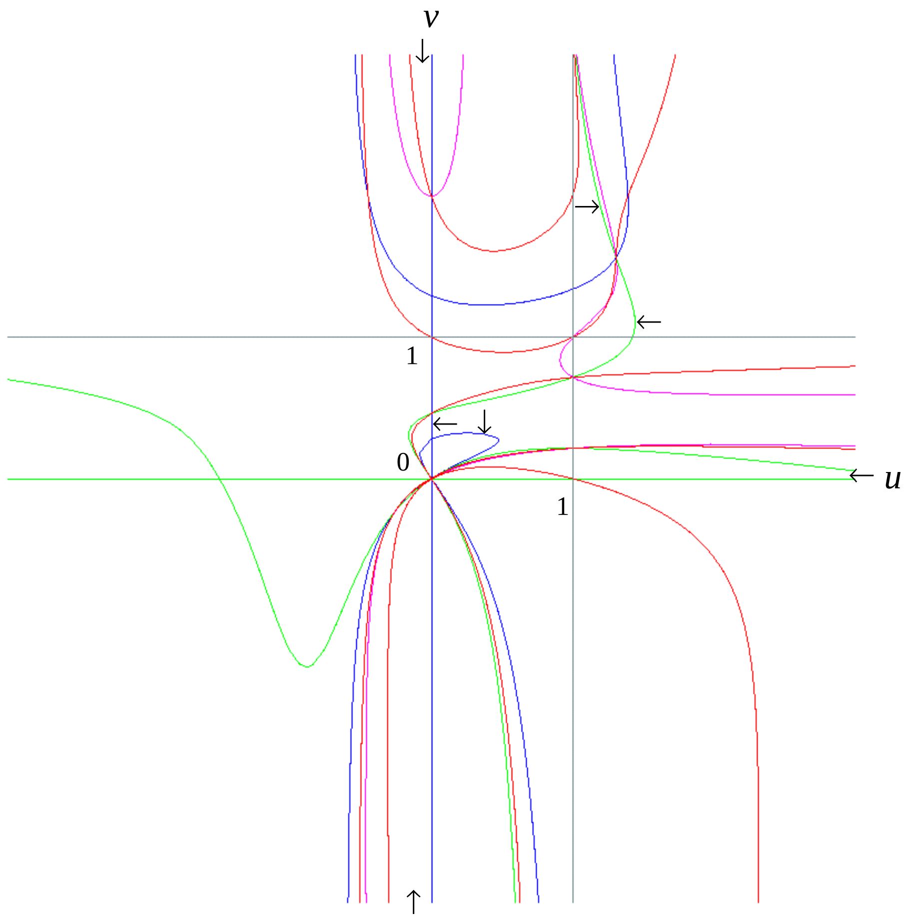


Fig. D.4: Parameter space for Example 4. Regions where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 1]$ are indicated by arrows. The curves are: blue — $\alpha_1 = 0$; green — $\alpha_2 = 0$; magenta — $\alpha_3 = 0$; red — $\alpha_4 = 0$; gray — $x = 1$ and $y = 1$.