M 340L – CS
Homework Set 8 Solutions

1. For a fixed value of k, let $L_k = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & \cdots & l_{k+1,k} & 1 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & l_{n,k} & 0 & \cdots & 1 \end{bmatrix}$. To be precise the *i,j* component of L_k is 0 **except**

(1). if $i = j$ the component is 1 (i.e. this is the diagonal)
and
(2). if $i = k+1,...,n$ and $j = k$ the component is $l_{i,k}$ which may be non-zero (this is below the diagonal in the k^{th} column).

a. The effect of multiplying L_k times a vector x is to add $l_{i,k}$ times x_k to x_i for $i=k+1, \ldots, n$.

b. The effect of multiplying L_k times a matrix A is to add $l_{i,k}$ times row k to row i for $i=k+1, \ldots, n$.

c. The inverse of L_k , $L_k^{-1} = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & \cdots & -l_{k+1,k} & 1 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -l_{n,k} & 0 & \cdots & 1 \end{bmatrix}$.

d. Let $c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{n,k} \end{bmatrix}$ thus 0's are in the first k positions. Let $e_k^T = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$, a row vector with all zeros except for a 1 in position k. What is the matrix $I + ce_k^T$? (Remember a column times a row is an outer product and results in a matrix.)

The matrix $I + ce_k^T = L_k$.

2. Let P_{ij} be the identity matrix with rows *i* and *j* swapped. To be precise the *p,q* component of P_{ij} is 0 **except**

(1). if $p = q$ but are **not** equal to *i* or *j* the component is 1 (i.e. the diagonal)
and
(2). The *i,j* and the *j,i* components are 1.

a. The effect of multiplying P_{ij} times a vector x is to swap components *i* and *j*.

b. The effect of multiplying P_{ij} times a matrix A is to swap rows *i* and *j*.

c. The inverse of P_{ij} , P_{ij}^{-1} is P_{ij}.

3. **Rank One Changes to *I*.** An outer product uv^T for column vector u and row vector v^T is sometimes called a *rank one matrix* since there is only one linearly independent column (and one linearly independent row). By adding this to the identity matrix, thus forming $\boldsymbol{I} + uv^T$, some interesting things happen. We call a matrix of the form $I + uv^T$ a *rank one change to I.*

You never actually compute the matrix $I + uv^T$: that's a waste of time and storage. To see this answer these questions:

a. Knowing that a matrix is a rank one change to *I*, (i.e. is $I + uv^T$, for a given *u* and *v*) what do you need to store?

Only *u* and *v* are stored.

b. Given a vector x, how would you actually compute $(I + uv^T)x$? I am asking for an algebraic expression here that equals $(I + uv^T)x$ but looks much more efficient. (Hint: remember the associativity of matrix products and that scalars are easy to multiply.)

$(I + uv^T)x$ is computed as $x + (v^T x)u$.

c. Using part b. and assuming both u and v^T are *n*-vectors, how many multiplications and additions are required to compute $(I + uv^T)x$?

This requires approximately *2n* multiplications and *2n* additions.

d. For this part only, assume that $v^T = e_k^T$ (from problem 1d). How many multiplications and additions are required to compute $(I + ue_k^T)x$?

This requires approximately *n* multiplications and *n* additions since no inner product is done..

d. Find the inverse of $I + uv^T$. (**Giant Hint**: Try something of the form $I + \alpha uv^T$ for a properly defined α. Remember the inverse of any invertible matrix A satisfies $AA^{-1} = I$. That should help you find α.)

We have $I = (I + uv^T)(I + \alpha uv^T) = I + uv^T + \alpha uv^T + uv^T \alpha uv^T = I + (1 + \alpha + \alpha v^T u)uv^T$ so $0 = 1 + \alpha + \alpha v^T u$ thus $\alpha = \dfrac{-1}{1 + v^T u}$.

e. Having done that, can you give some condition on u and v^T such that there is no inverse?

An inverse exists if and only if $v^T u \neq -1$.

4. Answer true or false to the following. Assume all matrices are $n \times n$. If false offer a counterexample.

a. If there is an $n \times n$ matrix D such that $AD = I$, then $DA = I$.

True. If $AD = I$, then $D = A^{-1}$ and $DA = A^{-1}A = I$.

b. If the linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then A is non-singular.

False. The transformation $x \mapsto 0x$ maps $\mathbb{R}^1$ into $\mathbb{R}^1$, but $A = 0$ is singular.

c. If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^n$.

True. According to the "17 equivalencies of nonsingularity" if the columns of A are linearly independent the column space of A is $\mathbb{R}^n$.

d. If the equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$, then the transformation $x \mapsto Ax$ is not one-to-one.

False. The equation $x = b$ has at least one solution for each b in $\mathbb{R}^n$, but the transformation $x \mapsto x$ *is* one-to-one

e. If there is a b in $\mathbb{R}^n$ such that the equation $Ax = b$ is consistent, then the solution is unique.

False. The 1×1 equation $0x = 0$ is consistent but has solutions $x = 0$ and $x = 1$.

5. When is a square lower triangular matrix invertible?

A square lower triangular matrix invertible if and only if all diagonal components are non-zero.

6. If an $n \times n$ matrix A is invertible, then the columns of A^T are linearly independent. Explain why.

According to the "17 equivalencies of nonsingularity" if A is invertible then A^T is also invertible and thus has linearly independent columns. (Alternatively, if A is invertible then it has linearly independent rows, but these rows are the columns of A^T, so A^T has linearly independent columns.

7. Can a square matrix with two identical rows be invertible? Why or why not?

If two rows are identical then they are linearly dependent thus the matrix is not invertible.

8. If A is a 5×5 matrix and the equation $Ax = b$ is consistent for every b in $\mathbb{R}^5$; is it possible that for some b, the equation $Ax = b$ has more than one solution? Why or why not?

If A is a square matrix and the equation $Ax = b$ is consistent for every b then the matrix is invertible and according to the "17 equivalencies of nonsingularity" the solution is unique.

9. If $n \times n$ matrices E and F have the property that $EF = I$; then E and F commute (i.e., . $EF = FE$). Explain why.

If square matrices E and F have the property that $EF = I$ then $E = F^{-1}$ and $EF = I = FF^{-1} = FE,$ so E and F commute.

10. Let A and B be $n \times n$ matrices. Show that if AB is invertible, so is B.

Suppose that B is not invertible, then for some $x \neq 0,\ Bx = 0,$ but then $ABx = 0$ which contradicts the fact that AB is invertible. Thus, B is invertible.

11. Suppose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(u) = T(v)$ for some pair of distinct vectors u and v in $\mathbb{R}^n$. Can T map $\mathbb{R}^n$ onto $\mathbb{R}^n$? Why or why not?

According to the "17 equivalencies of nonsingularity" if the transformation is not one-to-one then the linear system $T(x) = b$ has no solution x for some choice of $b.$ Thus, T does *not* map $\mathbb{R}^n$ onto $\mathbb{R}^n$.