M 340L – CS
Homework Set 11 Solutions

1. Calculate the determinants of

a. $\begin{bmatrix} 3 & 6 \\ -1 & 4 \end{bmatrix}$, $\det(\begin{bmatrix} 3 & 6 \\ -1 & 4 \end{bmatrix}) = 3\cdot 4 - 6\cdot(-1) = 18.$

b. $\begin{bmatrix} 2 & 2 & 4 \\ -2 & 0 & 3 \\ 4 & 3 & -1 \end{bmatrix}$, $\det(\begin{bmatrix} 2 & 2 & 4 \\ -2 & 0 & 3 \\ 4 & 3 & -1 \end{bmatrix})$

$= 2\cdot 0\cdot(-1) - 2\cdot 3\cdot 3 - (-2)\cdot 2\cdot(-1) + (-2)\cdot 3\cdot 4 + 4\cdot 2\cdot 3 - 4\cdot 0\cdot 4 = -22.$

2. The expansion of a 3 x 3 determinant can be remembered by the following device. Add a copy of the first two columns to the right of the matrix, and compute the determinant adding the products along the northwest-to-southeast diagonals and subtracting the products along the northeast-to-southwest diagonals:

$$\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix} - \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

Use this method to compute the determinants:

a. $\begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix}$, $\det(\begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix})$

$= 0\cdot(-3)\cdot 1 + 5\cdot 0\cdot 2 + 1\cdot 4\cdot 4 - 1\cdot(-3)\cdot 2 - 0\cdot 0\cdot 4 - 5\cdot 4\cdot 1 = 2.$

b. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$, $\det(\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix})$

$= 1\cdot 1\cdot 2 + 3\cdot 1\cdot 3 + 5\cdot 2\cdot 4 - 5\cdot 1\cdot 3 - 1\cdot 1\cdot 4 - 3\cdot 2\cdot 2 = 20.$

3. Prove that for an invertible matrix $A, \det(A^{-1}) = 1/\det(A)$. (Hint: Remember $AA^{-1} = I$.)

$1 = \det(I) = \det(AA^{-1}) = \det(A)\cdot\det(A^{-1})$, so $\det(A^{-1}) = 1/\det(A)$.

4. Answer true or false to the following. If false offer a counterexample.

a. If the columns of A are linearly dependent, then $\det(A) = 0$.

True. If the columns of A are linearly dependent, then A is singular and $\det(A) = 0$.

b. $\det(A+B) = \det(A)\det(B)$.

False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ then

$\det(A+B) = \det(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 1 \neq 0 = 0 \cdot 0 = \det(A)\det(B)$.

c. If two row interchanges are made in succession, then the new determinant equals the old determinant.

True. Let $B = P_1 P_2 A$, where P_1 and P_2 are actual swaps. Since $\det(P_1) = \det(P_2) = -1$, $\det(B) = \det(P_1 P_2 A) = \det(P_1)\det(P_2)\det(A) = (-1)(-1)\det(A) = \det(A)$.

d. The determinant of A is the product of the diagonal entries in A.

False. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\det(A) = -1 \neq 0 \cdot 0$.

e. If $\det(A)$ is zero, then two rows or two columns are the same, or a row or a column is zero.

False. Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$, then $\det(A) = 0$ but no two rows nor two columns are the same, nor is a row or a column zero.

5. Answer true or false to the following. If false offer a counterexample.

a. If $Ax = \lambda x$ for some scalar λ, then x is an eigenvector of A.

False. Let $A = [1], x = [0]$ then $Ax = 0 = 0 \cdot 0 = 0x$ but $x = [0]$ is not an eigenvector of A.

b. If $v1$ and $v2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $Av1 = 1v1$ and $Av2 = 1v2$, so both $v1$ and $v2$ are linearly independent eigenvectors with the common eigenvalue 1.

c. The eigenvalues of a matrix are on its main diagonal.

False. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $v1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then $Av1 = 1v1$ and $Av2 = -1v2$, so the eigenvalues are 1 and -1, neither ow which is on the diagonal.