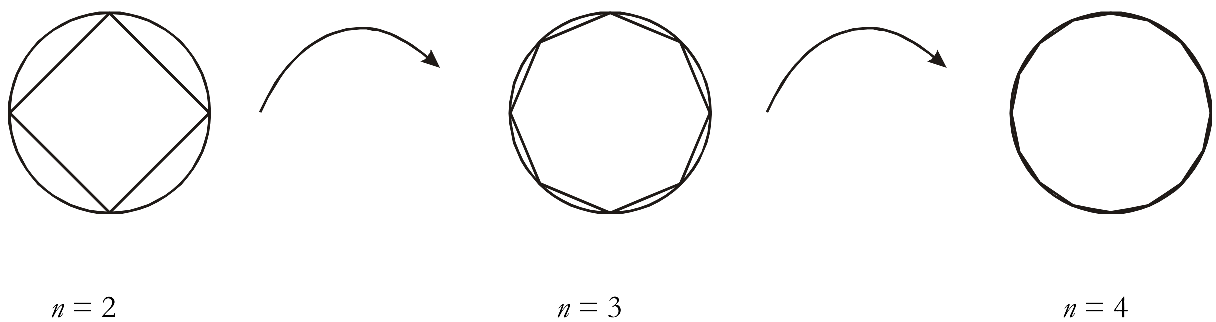
Determination of π
Due Tuesday Sept. 24 by 9:30 AM

The purpose of this problem is to get experience with floating point computation by writing a short section of code to compute a sequence of approximations to π.

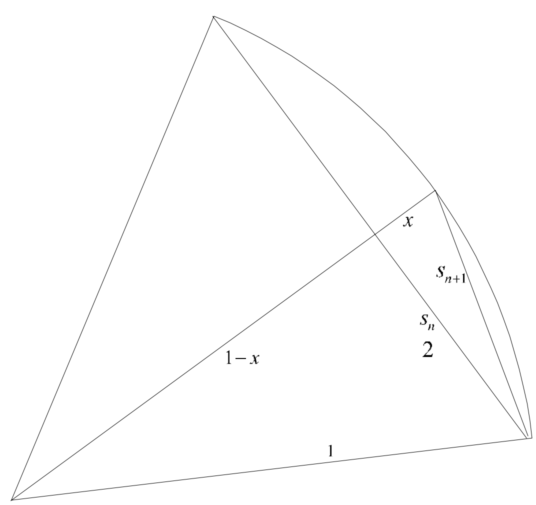
Model: We recognize π as the ratio of the circumference of a circle to its diameter. We can approximate the circumference of the circle by the perimeter of regular polygons by an increasingly larger number of sides. Since these perimeters converge to the circumference of the circle, the ratios of the perimeters to the limiting circle's diameter should provide a sequence with limit equal to π. Assume the circle has a radius of one (i.e., a "unit circle") and thus a diameter of 2. Let p_n represent the perimeter of a regular polygon with 2^n sides that is inscribed in the unit circle. We claim:

$$\lim_{n\to\infty} \frac{p_n}{2} = \pi$$



Analysis: We know that if we can determine the length of a side of the regular polygon of 2^n sides that is inscribed in the unit circle (call this length s_n) then $p_n = 2^n s_n$. Simple geometry for the inscribed square tells us that $s_2 = \sqrt{2}$. Our idea is to use the value of s_2 to determine s_3, the side of the inscribed regular octagon, then to use that to get s_4, the side of the inscribed regular hexadecagon, and so on. The general step is to use s_n to obtain s_{n+1}.

Consider the following depiction of the inscribed polygon with 2^n sides. The side of the polygon of 2^{n+1} sides is indicated and is obtained by bisecting the arc subtended by the side of length s_n:



Noticing the two right triangles and applying the Pythagorean Theorem twice, we have:

$$(1-x)^2 + \left(\frac{s_n}{2}\right)^2 = 1$$

and

$$x^2 + \left(\frac{s_n}{2}\right)^2 = s_{n+1}^2$$

Solving for x in the first equation, we have

$$x = 1 - \sqrt{1 - \left(\frac{s_n}{2}\right)^2}$$

Thus

$$x^2 = 1 - 2\sqrt{1 - \left(\frac{s_n}{2}\right)^2} + \left(1 - \left(\frac{s_n}{2}\right)^2\right)$$

$$= 2 - 2\sqrt{1 - \left(\frac{s_n}{2}\right)^2} - \left(\frac{s_n}{2}\right)^2$$

and by substituting this into the second equation above, we finally obtain:

$$2 - 2\sqrt{1 - \left(\frac{s_n}{2}\right)^2} = s_{n+1}^2$$

or

$$s_{n+1} = \sqrt{2\left(1 - \sqrt{1 - \left(\frac{s_n}{2}\right)^2}\right)}$$

Remembering that $p_n = 2^n s_n$ and thus $p_{n+1} = 2^{n+1} s_{n+1}$, we can remove the variables representing the sides and express everything in terms of the polygon perimeters:

$$\frac{p_{n+1}}{2^{n+1}} = \sqrt{2\left(1 - \sqrt{1 - \left(\frac{p_n}{2 \cdot 2^n}\right)^2}\right)}$$

that is

$$\frac{p_{n+1}}{2} = 2^n \sqrt{2\left(1 - \sqrt{1 - \left(\frac{p_n}{2 \cdot 2^n}\right)^2}\right)}$$

and recalling that $\frac{p_n}{2}$ for $n = 2,3,\ldots$ is the sequence of quantities that converges to π, let is use the label P_n for this approximation to π, and finally we obtain the recurrence relation that

$$P_{n+1} = 2^n \sqrt{2\left(1 - \sqrt{1 - \left(\frac{P_n}{2^n}\right)^2}\right)}$$

for $n = 2,3,\ldots$

which we start with $P_2 = \frac{p_2}{2} = \frac{4s_2}{2} = 2\sqrt{2}$, which is the approximation to π based upon the inscribed square.

Problem: Write a program of code that uses the recurrence above to produce $P_2, P_3, \ldots, P_{40}$. Make sure you use an output format that displays 15 digits.

Submission: Attach your M-file named picalc.m to a mail message to aniket@cs.utexas.edu.