Homework 11 Solutions
CS 336

1. Consider the set A of all finitely long strings of 0's and 1's. Prove that A is countably infinite.

> Consider $f: N \to A$ defined for $k = 0, 1, \ldots$ by $f(k) =$ the string formed from the binary representation of k+2 after the removal of the leading 1. That there is a leading 1 is guaranteed since $k + 2 \geq 2$. To show that this mapping is one-to-one, suppose i and j are two distinct natural numbers. Thus, the binary representations of i+2 and j+2 are also distinct. If these are distinct, so will be the strings remaining after the removal of the leading 1's. Thus $f(i) \neq f(j)$. To show that f is onto consider any such string s. Interpret the string formed by concatenating 1 with s as a binary number, subtract 2 from this number, and call the result k. It is clear that $f(k) = s$. (An alternative proof uses the theorem that a countably infinite union of a collection of finite sets is countable. Since the set of strings of length n is finite, the union of all such sets is A. That A is infinite follows from observing that the mapping of appending a 0 to the end of every string maps A to a proper subset.)

2. Consider the set B of all finite subsets of integers. Prove that B is countably infinite.

> For natural numbers k let $\overline{B}_k = \{i \in Z \mid -k \leq i \leq k\}$ and B_k be all of the subsets of $\overline{B}_k$. Since $\overline{B}_k$ is finite, so will be B_k. But every finite subset of integers must be contained in some B_k (actually an infinite number of them). We conclude that $B = \bigcup_{k \in N} B_k$ and by Theorem 6, B is countable. A subset of B is $S = \{\{0\}, \{1\}, \{2\}, \ldots\}$ (the singletons of the natural numbers). That this subset is infinite is shown by Theorem 4 and the mapping $f: N \to S$ defined by $f(k) = \{k\}$ which is obviously one-to-one. We conclude that B is both countable and infinite, thus countable infinite.

3. Consider the set B of all integer-valued functions defined on the set $\{0,1\}$. Is B finite, countably infinite, or uncountably infinite? (For example, one such function is f(0) = -7, f(1) = 17) Prove your claim.

The set B is countably infinite. To prove this, first we define B_i as the set of all functions mapping 0 to i and 1 to an integer. Clearly $B = \bigcup_{i=-\infty}^{\infty} B_i$. If each set B_i is countably infinite then the theorem guarantees that B is countably infinite since it is the countably infinite union of countably infinite sets. To prove each B_i is countably infinite, let $g_i : N \to B_i$ be defined for $j = 0, 1, \ldots$ by $g_i(j) = f_j^i$ where $f_j^i(0) = i$ and $f_j^i(1) = j/2$ if j is even and $f_j^i(1) = -(j+1)/2$ if j is odd. This function g_j maps the natural numbers one-to-one onto B_i, thus B_i is countably infinite.