Asymptotic Dominance Theory

- **Definition 1**: Given the functions $f : \mathrm{N} \to R$ and $g : \mathrm{N} \to R$, f is *asymptotically dominated by* g if there exist non-negative constants M and N such that for all $n \geq N$, $|f(n)| \leq M|g(n)|$. This is denoted by $f = \mathrm{O}(g)$.

- **Definition 2**: Given the functions $f: N \to R$ and $g: N \to R$, $f = o(g)$ if for every positive ε, there exists a non-negative constant N such that for all $n \geq N$, $|f(n)| \leq \varepsilon |g(n)|$.

Theorem 1: If $f = \mathrm{O}(g)$, then for any constant *s*, $sf = \mathrm{O}(g)$.

Proof: By definition, there exist non-negative constants M and N such that for all $n \geq N$, $|f(n)| \leq M|g(n)|$. Thus for all $n \geq N$, $|sf(n)| \leq |s|M|g(n)|$. Therefore, $sf = \mathrm{O}(g)$. □

Theorem 2: If $f_1 = \mathrm{O}(g_1)$ and $f_2 = \mathrm{O}(g_2)$, then $f_1 + f_2 = \mathrm{O}(|g_1| + |g_2|)$.

Proof: By definition, there exist non-negative constants M_1 and N_1 such that for all $n \geq N_1$, $|f_1(n)| \leq M_1|g_1(n)|$ and there exist non-negative constants M_2 and N_2 such that for all $n \geq N_2$, $|f_2(n)| \leq M_2|g_2(n)|$. For $n \geq \max\{N_1, N_2\}$ both inequalities hold so $|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)| \leq M_1|g_1(n)| + M_2|g_2(n)| \leq \max\{M_1, M_2\}|g_1(n)| + |g_2(n)|$. Therefore, $f_1 + f_2 = \mathrm{O}(|g_1| + |g_2|)$. □

Corollary 2.1: If for $i = 1, 2, \ldots, k$, $f_i = \mathrm{O}(g_i)$, then $\sum_{i=1}^{k} f_i = \mathrm{O}(\sum_{i=1}^{k} |g_i|)$.

Theorem 3: If $f_1 = \mathrm{O}(g_1)$ and $f_2 = \mathrm{O}(g_2)$, then $f_1 + f_2 = \mathrm{O}(\max\{|g_1|, |g_2|\})$.

Proof: By definition, there exist non-negative constants M_1 and N_1 such that for all $n \geq N_1$, $|f_1(n)| \leq M_1|g_1(n)|$ and there exist non-negative constants M_2 and N_2 such that for all $n \geq N_2$, $|f_2(n)| \leq M_2|g_2(n)|$. For $n \geq \max\{N_1, N_2\}$ both inequalities hold so $|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)| \leq M_1|g_1(n)| + M_2|g_2(n)| \leq (M_1 + M_2)\max\{|g_1(n)| + |g_2(n)|\}$. Therefore, $f_1 + f_2 = \mathrm{O}(\max\{|g_1|, |g_2|\})$. □

Corollary 3.1: If for $i = 1, 2, \ldots, k$, $f_i = \mathrm{O}(g_i)$, then $\sum_{i=1}^{k} f_i = \mathrm{O}(\max_{i=1,\ldots,k} |g_i|)$.

Corollary 3.2: If for $i = 1, 2, \ldots, k$, $f_i = \mathrm{O}(g)$, then $\sum_{i=1}^{k} f_i = \mathrm{O}(g)$.

Theorem 4: If $f_1 = \mathrm{O}(g_1)$ and $f_2 = \mathrm{O}(g_2)$, then $f_1 \cdot f_2 = \mathrm{O}(g_1 \cdot g_2)$.

Proof: By definition, there exist non-negative constants M_1 and N_1 such that for all $n \geq N_1$, $|f_1(n)| \leq M_1 |g_1(n)|$ and there exist non-negative constants M_2 and N_2 such that for all $n \geq N_2$, $|f_2(n)| \leq M_2 |g_2(n)|$. For $n \geq \max\{N_1, N_2\}$ both inequalities hold so $|f_1(n) \cdot f_2(n)| = |f_1(n)| \cdot |f_2(n)| \leq M_1 |g_1(n)| \cdot M_2 |g_2(n)| \leq (M_1 \cdot M_2)\,(|g_1(n)| \cdot |g_2(n)|$. Therefore, $f_1 \cdot f_2 = \mathrm{O}(g_1 \cdot g_2)$. □

Corollary 4.1: If for $i = 1, 2, \ldots, k$, $f_i = \mathrm{O}(g_i)$, then $\prod_{i=1}^{k} f_i = \mathrm{O}(\prod_{i=1}^{k} g_i)$.

Theorem 5: If $f_1 = \mathrm{O}(g_1)$, $g_2 = \mathrm{O}(f_2)$, and g_2 has no zeros. then $f_1 / f_2 = \mathrm{O}(g_1 / g_2)$.

Proof: By definition, there exist non-negative constants M_1 and N_1 such that for all $n \geq N_1$, $|f_1(n)| \leq M_1 |g_1(n)|$ and there exist non-negative constants M_2 and N_2 such that for all $n \geq N_2$, $|g_2(n)| \leq M_2 |f_2(n)|$. Notice that since g_2 has no zeros, then neither does f_2. Inverting this inequality, we obtain that for all $n \geq N_2$, $|\,1/f_2(n)\,| \leq M_2 |\,1/g_2(n)\,|$. For $n \geq \max\{N_1, N_2\}$ both inequalities hold so $|f_1(n)/f_2(n)| = |f_1(n)| \cdot |\,1/f_2(n)\,| \leq M_1 |g_1(n)| \cdot M_2 |\,1/g_2(n)\,| \leq (M_1 \cdot M_2)\,(|g_1(n)| / |g_2(n)|$. Therefore, $f_1 / f_2 = \mathrm{O}(g_1 / g_2)$. □

Theorem 6: If $a \leq b$, then $n^a = \mathrm{O}(n^b)$

Proof: For $n \geq 0$, $n^{-(b-a)} \leq n^0 = 1$, and $|n^a| = |n^{-(b-a)}\, n^b| = |n^{-(b-a)}|\,|n^b| \leq 1 \cdot |n^b|$. Therefore, $n^a = \mathrm{O}(n^b)$. □

Theorem 7: If $a < b$, then $n^a = o\,(n^b)$

Proof: Given any $\varepsilon > 0$, let $N = (1/\varepsilon)^{1/(b-a)}$. Notice then for $n \geq N = (1/\varepsilon)^{1/(b-a)}$, $n^{b-a} \geq 1/\varepsilon$, and $n^{-(b-a)} \leq \varepsilon$. So $|n^a| = |n^{-(b-a)}\, n^b| = |n^{-(b-a)}|\,|n^b| \leq \varepsilon\, |n^b|$. Therefore, $n^a = o\,(n^b)$. □

Example 1: If $f_1 = O(g_1)$ and $f_2 = O(g_2)$, then f_1 / f_2 may not be $O(g_1 / g_2)$.

Proof: Let $f_1(n) = f_2(n) = 1$, for all $n \geq 0$. Then $f_1 = O(1)$ and $f_2 = O(n)$ but $f_1 / f_2 = 1 \neq O(1/n)$. To see this, consider any $N \geq 0$ and $M \geq 0$. Choose any $n > \max\{N, M\}$. Notice that then $|1| = 1 > M / |n|$, so $1 \neq O(1/n)$. □

Example 2: If $a < b$, then $n^b \neq O(n^a)$.

Proof: Suppose $n^b = O(n^a)$, then there exist $N \geq 0$ and $M \geq 0$ so that for all $n \geq N$, $|n^b| \leq M|n^a|$. Choose any $n > \max\{N, M^{1/(b-a)}\}$. Notice that then $n^{b-a} > M$, so $|n^b| = n^b > M\, n^a = M\,|n^a|$, and $n^b \neq O(n^a)$. □

Example 3: If *f* is any polynomial of degree *k* then $f = O(n^k)$.

Proof-1: Without loss of generality, assume $f(n) = \sum_{i=0}^{k} a_i\, n^i$. For all $n \geq 0$ and $0 \leq i \leq k$, $|n^i| \leq |n^k|$ and $|a_i\, n^i| \leq |a_i|\,|n^k|$. So for $N = 0$ and $M = \sum_{i=0}^{k} |a_i|$, we have $n \geq N$ implies

$$|f(n)| = \left|\sum_{i=0}^{k} a_i\, n^i\right| \leq \sum_{i=0}^{k} |a_i\, n^i| \leq \sum_{i=0}^{k} |a_i|\,|n^k| \leq \left(\sum_{i=0}^{k} |a_i|\right) \cdot |n^k| = M\,|n^k|$$ □

Proof- 2: Without loss of generality, assume $f(n) = \sum_{i=0}^{k} a_i\, n^i$. By Theorem 6, $n^i = O(n^k)$, for $0 \leq i \leq k$. By Theorem 1, $a_i\, n^i = O(n^k)$ for $0 \leq i \leq k$. Finally, from Corollary 3.2, $f(n) = \sum_{i=0}^{k} a_i\, n^i = O(n^k)$. □