Aids to the Proving of Asymptotic Dominance

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1. Proving $f = \mathrm{O}(g)$.

The definition of $f = \mathrm{O}(g)$ requires:

> Given the functions $f : \mathrm{N} \to R$ and $g : \mathrm{N} \to R$, $f = \mathrm{O}(g)$ if there exist non-negative constants M and N such that for all $n \geq N$, $|f(n)| \leq M|g(n)|$.

The requirement that $|f(n)| \leq M|g(n)|$ is equivalent to $\left|\frac{f(n)}{g(n)}\right| \leq M$ if $g(n) \neq 0$ (and thus the division by $g(n)$ is legal). In fact, if the function g has no zeros, then the definition can dispense with the constant N (i.e., $N = 0$ suffices). If g has a finite number of zeros, then showing $f = \mathrm{O}(g)$ is equivalent to showing that $\left|\frac{f(n)}{g(n)}\right|$ can be bounded from above for all values of n other than the where $g(n) = 0$. (If g has an infinite number of zeros, then showing $f = \mathrm{O}(g)$ can get a good deal more complicated. In fact, it does not hold unless for sufficiently large values of n, all of the zeros of g are also zeros of f.)

The following theorem is a good example to see how one might construct proofs of the form $f = \mathrm{O}(g)$ in general.

Theorem 6: If $0 \leq a \leq b$, then $n^a = \mathrm{O}(n^b)$

Discovery of proof: We want to find the M and N such that for all $n \geq N$, $|n^a| \leq M|n^b|$.

> **Hint #1:** When proving $f = \mathrm{O}(g)$, choose N big enough to avoid all zeros in g and also big enough to avoid any unusual behavior in both f and g; then forget about N and concentrate on M.

Consistent with the hint, we'll take $N = 1$ and then concentrate on M. In the end we'll need $|n^a| \leq M|n^b|$.

> **Hint #2:** When proving $f = \mathrm{O}(g)$, after choosing N, see if you can bound $\left|\frac{f(n)}{g(n)}\right|$ for all $n \geq N$. This bound will be M.

Using that hint, we need to bound $\left|\frac{n^a}{n^b}\right| = \frac{n^a}{n^b} = n^{a-b}$. But since $0 \le a \le b$, thus $a - b \le 0$, we can bound n^{a-b} by 1, so we take $M = 1$.

> **Hint #3:** When proving $f = \mathrm{O}(g)$, after choosing N and M in accordance with the previous hints, the format of the proof will be $|f(n)| = \left|\frac{f(n)}{g(n)}\right| |g(n)| \le M |g(n)|$ (although there may be some extra steps slipped into the chain of inequalities for clarity.

So in this case we will have $|n^a| = \left|\frac{n^a}{n^b} n^b\right| = |n^{a-b}| |n^b| \le 1 \cdot |n^b|$.

All of that was preliminary, but we can put it together to get the official proof.

Proof: For $n \ge 1$, $n^{a-b} \le n^0 = 1$, and $|n^a| = \left|\frac{n^a}{n^b} n^b\right| = |n^{a-b}| |n^b| \le 1 \cdot |n^b|$. Therefore, $n^a = \mathrm{O}(n^b)$. $\square$

2. Proving $f = o(g)$.

The definition of $f = o(g)$ requires:

Given the functions $f: N \to R$ and $g: N \to R$, $f = o(g)$ if for every positive ε, there exists a non-negative constant N such that for all $n \ge N$, $|f(n)| \le \varepsilon |g(n)|$.

The requirement that $|f(n)| \le \varepsilon |g(n)|$ for arbitrarily small ε is equivalent to $\lim_{n\to\infty} |f(n)/g(n)| = 0$ if one can ignore the zeros of g. (It is our intention here to discuss the techniques of formal proofs of results such as $f = o(g)$. Since it is assumed that the readers of this have not been acquainted with the formalities of limit proofs, transforming one problem into the other serves no benefits in terms of proofs.) Similar to the format used in the first section for proving $f = \mathrm{O}(g)$, in this section we will illustrate how $f = o(g)$ can be demonstrated. We will use an example.

Theorem 7: If $0 \le a < b$, then $n^a = o(n^b)$

Discovery of proof: This one is different from showing $f = \mathrm{O}(g)$. Here we want to assume we are given a positive ε, and then try to discover an N such that for all $n \ge N$, $|n^a| \le \varepsilon |n^b|$. (By the way, notice that the hypothesis here insists that a is strictly less that b. If not for that this result would not be true.)

Hint #4: When proving $f = o(g)$, start with a temporary N (call it $\bar{N}$) big enough to avoid all zeros in g and also big enough to avoid any unusual behavior in both f and g.

This is easy in this example: chose $\bar{N} = 1$. Later we will make sure that our N is at least this big by doing the max trick. Henceforth, however, we will be assuming at least that $n \geq \bar{N}$.

Hint #5: When proving $f = o(g)$, (assuming you are out of the range of any zeros of g) consider the inequality $\left|\frac{f(n)}{g(n)}\right| \leq \varepsilon$, which is equivalent to $|f(n)| \leq \varepsilon |g(n)|$. Try to manipulate this into the form $n \geq \mathit{something}$. The *something* may depend upon ε but it may not involve n.

So in this case we want to manipulate the inequality $\left|\frac{n^a}{n^b}\right| \leq \varepsilon$. We could have a chain of inequalities such as this:

$$\left|\frac{n^a}{n^b}\right| \leq \varepsilon$$

$$n^{a-b} \leq \varepsilon$$

$$n^{b-a} = n^{-(a-b)} \geq \varepsilon^{-1} = 1/\varepsilon$$

$$n \geq (1/\varepsilon)^{1/(b-a)}$$

So our $\mathit{something} = (1/\varepsilon)^{1/(b-a)}$.

Caution: Try to construct the chain of inequalities in such a fashion that the order can be reversed. Your proof will essentially be this reveral. Notice that $n \leq 3/\varepsilon$ guarantees that $n \leq 4/\varepsilon$ but the reverse does not work.

Notice that someplace in the chain of inequalities the inequality generally gets turned backwards. Typically this happens by inversion or multiplication by a negative number. Here are some general rules.

Reminders on Inequalities: Multiplication by non-negative number maintains inequalities, so $x \le y \Rightarrow sx \le sy$ if $s \ge 0$, but if $s < 0$ the inequality is reversed to $sx \ge sy$. Inversion of positive quantities reverses inequalities, so $0 < x \le y \Rightarrow \frac{1}{x} \ge \frac{1}{y}$. The general implication of $x \le y \Rightarrow h(x) \le h(x)$ holds only for *monotonic* functions h. Examples of monotonic functions are $h(x) = c^x$ for any $c > 1$, $h(x) = \log_c x$ for any $c > 1$, $h(x) = x^c$ for any $c > 0$, and $h(x) = \sqrt[c]{x} = x^{1/c}$ for any $c > 1$. Remember, using monotonic functions maintains inequalities.

Hint #6: When proving $f = o(g)$, after choosing $\bar{N}$ and *something* according to the preceding hints, let $N = \max\{\bar{N}, \textit{something}\}$. By doing this "max trick", taking $n \ge N$ guarantees that both $n \ge \bar{N}$ and $n \ge \textit{something}$. Now reverse the $n \ge \textit{something}$ inequality back to $\left|\frac{f(n)}{g(n)}\right| \le \varepsilon$ and then conclude by using the equivalent form $|f(n)| \le \varepsilon |g(n)|$. You have shown $f = o(g)$.

Recall that in this example our $\bar{N} = 1$ and $\textit{something} = (1/\varepsilon)^{1/(b-a)}$, so we will let $N = \max\{1, (1/\varepsilon)^{1/(b-a)}\}$. The proof will just work backward from $n \ge (1/\varepsilon)^{1/(b-a)}$ to $|n^a| \le \varepsilon\ |n^b|$. Here it is.

Proof: Given any $\varepsilon > 0$, let $N = \max\{1, (1/\varepsilon)^{1/(b-a)}\}$. Notice then for $n \ge N$, $n \ge (1/\varepsilon)^{1/(b-a)}$, so $n^{b-a} \ge 1/\varepsilon$, and $n^{-(b-a)} \le \varepsilon$. Therefore, $|n^a| = |n^{-(b-a)}\ n^b| = |n^{-(b-a)}|\,|n^b| \le \varepsilon\ |n^b|$ and $n^a = o(n^b)$. □

3. Proving $f \ne \mathrm{O}(g)$.

Now we can turn to some ideas for proving a result of the form $f \ne \mathrm{O}(g)$. If we negate the definition of $f = \mathrm{O}(g)$ we obtain:

> Given the functions $f : \mathrm{N} \to R$ and $g : \mathrm{N} \to R$, $f \ne \mathrm{O}(g)$ if, for all non-negative constants M and N, there exists an $n \ge N$ so that $|f(n)| > M\,|g(n)|$.

Showing $f \ne \mathrm{O}(g)$ can be tricky because, by definition, the proof must consider any given non-negative constants M and N. In the end, however, we only have to find a single

$n \geq N$ such that $|f(n)| > M|g(n)|$. Recall that when we were showing $f = \mathrm{O}(g)$, it was observed that this was equivalent to showing $\left|\frac{f(n)}{g(n)}\right| \leq M$ (if we are able to ignore zeros of g). Now we want to show that no matter how large a bound M and how far out into the natural numbers is dictated by N, we can find an $n \geq N$ such that $\left|\frac{f(n)}{g(n)}\right| > M$. Consider this example.

Example: If $0 < a < b$, then $b^n \neq \mathrm{O}(a^n)$.

Discovery of proof: We assume we are given non-negative constants M and N. The task is to find the $n \geq N$ so that $|b^n| > M|a^n|$.

> **Hint #7:** When proving $f \neq \mathrm{O}(g)$, start with a temporary N (call it $\bar{N}$) big enough to avoid all zeros in g and also big enough to avoid any unusual behavior in both f and g.

For this particular problem we don't really need to worry about this. We could simply take $\bar{N} = 0$. Since our functions are defined only for natural numbers, as will be seen, a zero value of $\bar{N}$ adds nothing to the assumptions. The key issue is making $|b^n| > M|a^n|$ or, equivalently, making $\left|\frac{b^n}{a^n}\right| > M$.

> **Hint #8:** When proving $f \neq \mathrm{O}(g)$, (assuming you are out of the range of any zeros of g) consider the inequality $\left|\frac{f(n)}{g(n)}\right| > M$, which is equivalent to $|f(n)| > M|g(n)|$. Try to manipulate this into the form $n > \mathit{something}$. The *something* may depend upon M but it may not involve n.

In our example here, we want $\left|\frac{b^n}{a^n}\right| > M$. We could have a chain of inequalities such as this:

$$\left|\frac{b^n}{a^n}\right| > M$$

$$(\frac{b}{a})^n > M$$

$$n\log\frac{b}{a} > \log M$$

$$n > \frac{\log M}{\log\frac{b}{a}}$$

So our $something = \dfrac{\log M}{\log\frac{b}{a}}$.

> **Hint #9:** When proving $f \neq O(g)$, after choosing $\bar{N}$ and $something$ according to the preceding hints, let $N = \max\{\bar{N}, something + 1\}$. By doing this "max trick", taking $n \geq N$ guarantees that both $n \geq \bar{N}$ and $n > something$. Now reverse the $n > something$ inequality back to $\left|\dfrac{f(n)}{g(n)}\right| > M$ and then conclude by using the equivalent form $|f(n)| = \left|\dfrac{f(n)}{g(n)}\right| |g(n)| > M|g(n)|$. You have shown $f \neq O(g)$.

Recall that in this example our $\bar{N} = 0$ and $something = \dfrac{\log M}{\log\frac{b}{a}}$, so we will let $N = \max\{0, \dfrac{\log M}{\log\frac{b}{a}} + 1\}$. The proof will just work backward from $n > \dfrac{\log M}{\log\frac{b}{a}}$ to $b^n > M a^n$. Here it is.

Proof: Given $M \geq 0$ and $N \geq 0$, notice that $\ln\frac{b}{a} > 0$ and choose $n = \max\{N, \dfrac{\ln M}{\ln\frac{b}{a}} + 1\}$. For this *n,* we have $n \geq N$ and $n > \dfrac{\ln M}{\ln\frac{b}{a}}$, thus $n\ln\frac{b}{a} > \ln M$ and $(\frac{b}{a})^n > M$. But then $|b^n| = b^n > M\, a^n = M\,|a^n|$ so $b^n \neq O(a^n)$. □

4. Proving $f \neq o(g)$.

If we negate the definition of $f = o(g)$ we get:

> Given the functions $f: N \to R$ and $g: N \to R$, $f \neq o(g)$ if there exists a positive ε, such that for every non-negative constant N there exists an $n \geq N$, such that $|f(n)| > \varepsilon\,|g(n)|$.

In this case we are allowed to choose the ε but that choice can be tricky. Recall it was mentioned in Section 2 that another interpretation of $f = o(g)$ was that $\lim_{n\to\infty} |f(n)/g(n)| = 0$. If $\lim_{n\to\infty} |f(n)/g(n)|$ exists but is positive, then choosing ε to be anything less than this limit (e.g., half the limit) will work for showing $f \neq o(g)$. Even if the limit does not exist, if there is an infinite number of values of $|f(n)/g(n)|$ greater than some positive quantity δ, then choosing ε to be anything less δ will work. As in the previous sections, we will illustrate how $f \neq o(g)$ can be demonstrated through an example.

Example: If $a, b > 0$, then $\log_a n \neq o \log_b n$[1].

Discovery of proof: The most important task is to determine the appropriate ε. The value of $\lim_{n\to\infty} |\log_a n / \log_b n|$ is obviously $|\log_a b|$ since $\log_a n = \log_b a \cdot \log_a n$ for $n \geq 1$.

> **Hint #10:** When proving $f \neq o(g)$, consider the ratio $|f(n)/g(n)|$ for large values of n. If this ratio approaches some value $L > 0$ for any infinite subset of the natural numbers, select $\varepsilon = L/2$.

As an example, if just for prime numbers n, the ratio $|f(n)/g(n)|$ tends to some positive L, that L will work. We know that for $f = o(g)$, the ratio would have to tend to zero, so if an infinite subset of the natural exists for which the ratio tends to a non-zero (hence positive) value, we could not have that $f = o(g)$.

In the particular example we know that the limit of the ratio is $|\log_a b|$, so we will let $\varepsilon = |\log_a b|/2$. The next step is discovering an n greater than or equal to a given N so that $|f(n)/g(n)| > \varepsilon$. This will then guarantee that $|f(n)| > \varepsilon\,|g(n)|$.

> **Hint #11:** When proving $f \neq o(g)$, (assuming you are out of the range of any zeros of g) consider the inequality $\left|\dfrac{f(n)}{g(n)}\right| > \varepsilon$, which is equivalent to $|f(n)| > \varepsilon |g(n)|$. Try to manipulate this into the form $n > \textit{something}$. The *something* may depend upon ε but it may not involve n.

For the example, we want $\left|\frac{\log_a n}{\log_b n}\right| > |\log_a b|/2$. In general, we would have a chain of inequalities transforming this into $n > \mathit{something}$, but since for $n \ge 1$ $\left|\frac{\log_a n}{\log_b n}\right| = |\log_a b| > 0$, it is obvious that for any $n \ge 1$, $\left|\frac{\log_a n}{\log_b n}\right| > |\log_a b|/2$. Our *something* here doesn't even depend upon ε (but typically it would). We can take $\mathit{something} = 0$.

> **Hint #12:** When proving $f \ne o(g)$, after choosing *something* according to the preceding hint, let $n = \max\{N, \mathit{something}+1\}$. Taking this n guarantees that both $n \ge N$ and $n > \mathit{something}$. Now reverse the $n > \mathit{something}$ inequality back to $\left|\frac{f(n)}{g(n)}\right| > \varepsilon$ and then conclude by using the equivalent form $|f(n)| = \left|\frac{f(n)}{g(n)}\right||g(n)| > \varepsilon|g(n)|$. You have shown $f \ne o(g)$.

We are ready for the proof.

Proof: Let $\varepsilon = |\log_a b|/2$. Given any non-negative N, let $n = \max\{N, 1\}$. Since $\log_a n = \log_b a \cdot \log_a n$, we have $|\log_a n| = |\log_b a||\log_b n| > |\log_b a|/2|\log_b n| = \varepsilon|\log_b n|$. We conclude that $f \ne o(g)$. $\square$