1	5
2	15
3	20
4	10
5	15
6	15
Total	80

Examination 1 Solutions
CS 336

1. [5] Given sets A and B, each of cardinality $n \geq 1$, how many functions map A in a one-to-one fashion onto B?

Let $A = \{a_1, a_2, ..., a_n\}$ and $f : A \xrightarrow[onto]{1-1} B$. There are n options for the value of $f(a_1)$ and, given that, $n-1$ options for the value of $f(a_2)$, ..., and one option for the value of $f(a_n)$. Since B also has cardinality n this function is automatically onto. Thus, there are $n!$ such one-to-one and onto functions.

2 a. [5] Given the set of r symbols $\{a_1, a_2, ..., a_r\}$, how many different strings of length $n \geq 1$ exist (allowing repetitions)?

For each of n positions there are r options. Thus, there are r^n such strings.

b. [10] Given the set of r symbols $\{a_1, a_2, ..., a_r\}$, how many different strings of length $n \geq 2$ exist that contain at least one a_1 and at least one a_2? (Assume $r \geq 2$.)

There are $(r-1)^n$ strings that avoid the element a_1 and the same number that avoid a_2. There are $(r-2)^n$ subsets that avoid both elements so there are $2(r-1)^n - (r-2)^n$ strings that either avoid a_1 or avoid a_2. Considering the complement, there are $r^n - 2(r-1)^n + (r-2)^n$ strings that contain at least one a_1 and at least one a_2.

3. [10] Present a combinatorial argument that for all positive integers n :

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k .$$

Consider as a model strings of length n using the characters from the set $\{a, b, c\}$. For each n positions there are 3 options so there are 3^n such strings. Alternatively, let k represent the number of positions in the string not occupied by a (i.e., thus, occupied by either b or c). The value of k can vary between 0 and n. For a fixed number k of b s and c s, there are $\binom{n}{k}$ ways to determine the positions to be occupied by the b s and c s and then 2 choices (either b or c) for each of these k positions, for a total of $\binom{n}{k}2^k$ possibilities. The remaining $n-k$ positions must be occupied by a s. Summing over all possible values of k. We have $\sum_{k=0}^{n}\binom{n}{k}2^k$ such strings and this must equal 3^n.

b. [10] Present a combinatorial argument that for all integers $n \geq 3$:

$$\binom{3n}{3} = 3\binom{n}{3} + 3\cdot 2n\binom{n}{2} + n^3$$

(**Hint:** Consider three pairwise disjoint sets of cardinality n.)

Let $A, B,$ and C be pairwise disjoint sets of cardinality n. Consider as a model the number of subsets of $A \cup B \cup C$ of cardinality 3. Since the cardinality of $A \cup B \cup C$ is $3n$, there are $\binom{3n}{3}$ such subsets of cardinality 3. Now consider that all three elements could come from the same set $A, B,$ or C, that two could come from one and one comes from another, and that each of the three could come from a different set. In the first case, there are 3 options for the set and then $\binom{n}{3}$ ways of selecting the subset. In the second case, there are there are 3 options for the set from which 2 elements are selected, then $\binom{n}{2}$ ways of selecting those two elements, and 2 choices for the set from which only one element is selected, and finally n options for that selection. In the final case, there are n possible selections from each of the three sets. The total is $3\binom{n}{3} + 3\cdot 2n\binom{n}{2} + n^3$ and this must equal $\binom{3n}{3}$.

4. [10] A multiset is similar to a set in that order is irrelevant but multiple copies of elements are allowed. For example, the **sets** $\{1,2,3\}$ and $\{1,1,1,2,2,3\}$ are identical and each has cardinality three but the **multisets** $\{1,2,3\}$ and $\{1,1,1,2,2,3\}$ are different and the first has cardinality three but the second has cardinality six. How many multisets of cardinality n are there that employ elements from $a_1, a_2, \ldots, a_r$?

Let us label r bins $a_1, a_2, \ldots, a_r$ and consider the number of ways of placing n indistinguishable balls into the r bins. The placements of balls into bins is in one-to-one correspondence with multisets of cardinality n that employ elements from

$a_1, a_2, \ldots, a_r$. There are $\binom{n+r-1}{n}$ such placements of balls in bins so there is the same number of multisets.

5 a. [5] How many strings are there of length $k \geq 1$ using elements from the set $\{1,2,\ldots,n\}$ if repetition is not allowed. (Assume $k \leq n$).

Since there are n options for the first element of the string, $n-1$ options for the second element of the strings, ..., and $n-(k-1)$ options for the last element there are $n \cdot (n-1) \cdot \cdots \cdot (n-(k-1)) = \frac{n!}{(n-k)!}$ such strings.

b. [10] Now assume each of the different strings in part **a.** is equally likely. What is the probability that the minimum of the *k* elements is less than or equal to *r*, where $1 \leq r \leq n-k+1$?

Now we must count how many of these strings have minimum of the *k* elements is less than or equal to *r.* Alternatively we could count how many of these strings have minimum of the *k* elements is strictly greater than *r.* If the minimum of the *k* elements is strictly greater than r, then there are only $n-r$ choices for first element of the string, $n-r-1$ options for the second element of the strings, ..., and $n-r-(k-1)$ options for the last element. So there are $(n-r) \cdot (n-r-1) \cdot \cdots \cdot (n-r-(k-1)) = \frac{(n-r)!}{(n-k)!}$ such strings. The probability of such a string is $\frac{(n-r)!}{n!}$ and the probability that the minimum of the *k* elements is less than or equal to *r* is $1 - \frac{(n-r)!}{n!}$.

6. a. [5] a. How many permutations of *a, b, c, d,* and *e* have both *a* to the left of *b* and *b* to the left of *c*?

There are 5 positions in the string for the *d* and then 4 positions for the *e*. Once these are fixed, the positions for *a*, *b*, and *c* are determined since *a* must be to the left of *b* and *b* to the left of *c* and these must fill the three remaining positions. Thus there are $5 \cdot 4 = 20$ such permutations.

b. [10] Assume all such permutations are equally likely, what is the probability that the permutation begins with *a* given that it has both *a* to the left of *b* and *b* to the left of *c*?

If the permutation begins with *a,* then there are only 4 positions in the string for the *d* and then 3 positions for the *e*. Thus there are 12 such strings with *a* in the first position and *b* to the left of *c (a* will automatically be to the left of *b).* The probability that the permutation begins with a given that it has both a to the left of b and b to the left of c is $12/20$.