Examination 1 Solutions

CS 336H

1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose the
$\boxed{\textbf{comments in boxes}}$

1. [5] Given m a's, n b's, and p c's, how many distinct sequences are there that employ each of the $m+n+p$ symbols? (Note: All a's are identical as are all b's and all c's)

The sequence is of length $m+n+p$ and is determined if the m positions for the a's are selected from the $m+n+p$ and then the n positions for the b's are selected from the remaining $n+p$. Thus there are $\binom{m+n+p}{m}\binom{n+p}{n} = \binom{m+n+p}{m \quad n}$ such sequences.

2 a. [5] How many permutations of a, b, c, d, e, and f have b to the left of c and e to the left of f?

There are $6!$ permutations of the six characters. In half of these b occurs to the left of c and in half of these b occurs to the right of c positions. Thus $6!/2$ have b to the left of c. With those fixed, in half of these e occurs to the left of f and in half of these e occurs to the right of f positions. Thus $6!/4$ permutations have b to the left of c and e to the left of f.

[5] b. How many permutations of a, b, c, d, e, and f have b to the left of c and d?

There are $6 \cdot 5 \cdot 4$ was to position the unrestricted characters a, e, and f. Of the three remaining positions, b must be in the leftmost and the other two can be either in the order c then d or d then c. Thus there are $6 \cdot 5 \cdot 4 \cdot 2 = 6!/3$ permutations.

3.a [10] Present a combinatorial argument that for all positive integers x and y

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n.$$

(Hint: Consider sequences drawn from the union of distinct sets A and B of cardinalities x and y, respectively.)

Let A and B be disjoint sets of cardinalities x and y, respectively and $C = A \cup B$. How many strings are there of length n using the characters of C . Let k represent the number of positions in the string occupied by elements of set A. The value of k may vary from 0 to n. There are $\binom{n}{k}$ such selections of the k positions and x choices for each element in the k positions. The remaining $n-k$ positions of the string must be occupied by elements from B, and the are y choices for each element in the $n-k$ positions. Thus, there are $\binom{n}{k} x^k y^{n-k}$ such stings for fixed k and $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ overall. Alternatively, we can think of having $x + y$ options for the character in each of the n positions for a total of $(x + y)^n$ such strings. This must equal $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

b [10] Present a combinatorial argument that for all positive integers $1 \le k \le m \le r$:

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

Consider a set A of cardinality r with a subset B of cardinality m and a subset C (of B) having k elements. How many such selections can be made for the elements of B and C?

There are $\binom{r}{m}$ selections for the subset B and then $\binom{m}{k}$ selections from those elements of B for the elements of C. Thus, there are $\binom{r}{m}\binom{m}{k}$ selections for the elements of B and C. Alternatively, we could first select the k elements of C from A and then select the $m-k$ elements of $B \sim C$ from $A \sim C$. This first can be done in $\binom{r}{k}$ ways and the second can be done in $\binom{r-k}{m-k}$ ways, so there are $\binom{r}{k}\binom{r-k}{m-k}$ selections for the elements of B and C and this must equal $\binom{r}{m}\binom{m}{k}$

4. [10] Consider 5-tuples of the form $<r_1, r_2, r_3, r_4, r_5>$, where the $r_i \geq 1$. How many such 5-tuples are there satisfying

$$r_1 + r_2 + r_3 + r_4 + r_5 = 20?$$

This is equivalent to placing 20 balls into five bins **except** that every bin must contain at least one ball. Thus five balls are fixed and the other 15 are free to distribute in any of the five bins. This can be done in $\binom{5+15-1}{15} = \binom{19}{15}$ ways.

5. [10] For $n \geq 5$, consider strings of length n using lower case roman letters (i.e., $\{a,b,c,\ldots,x,y,z\}$). Assuming all such strings are equally likely what is the probability that the string occurs in non-decreasing order (i.e. all *a*'s precede all *b*'s, all *b*'s precede all *c*'s ,... etc.)?

The are 26^n such equally likely strings. The number of these strings in which the characters occur in non-decreasing order is the same as the number of ways of placing n balls in 26 bins (since the strings is totally determined by knowing the number of *a*'s. *b*'s, etc.) There are $\binom{26+n-1}{n}$ such placements so the probability of a non-decreasing string is $\binom{26+n-1}{n} / 26^n$.

6. a. [10] For $n \geq 6$, consider strings of length n using elements of $\{a, b, c, d, e\}$. Assume all such strings are equally likely. What is the probability that a string has three or more *a*'s?

There are 5^n such equally likely strings. Of these, 4^n have no *a*., $\binom{n}{1} 4^{n-1}$ have exactly one *a*, and $\binom{n}{2} 4^{n-2}$ have exactly two *a*'s. Thus there are $5^n - 4^n - \binom{n}{1} 4^{n-1} - \binom{n}{2} 4^{n-2}$ strings with three or more *a*'s . The probability of such a sting is then $(5^n - 4^n - \binom{n}{1} 4^{n-1} - \binom{n}{2} 4^{n-2}) / 5^n$.

b. [5] What is the probability that such a string has exactly three *b*'s given that it has exactly three *a*'s?

There are $\binom{n}{3}4^{n-3}$ equally likely strings with exactly three *a*'s. There are $\binom{n}{3}\binom{n-3}{3}3^{n-6}$ strings having exactly three *b*'s and exactly three *a*'s. Thus the probability that such a string has exactly three *b*'s given that it has exactly three *a*' is $\binom{n}{3}4^{n-3} \Big/ \binom{n}{3}\binom{n-3}{3}3^{n-6}$.

7. [10] Consider strings of length $n \geq 2$ containing exactly k 1's and $n-k$ 0's and having no adjacent 1's (i.e., there is at least one 0 between any 1's). Assuming $k \geq 1$ and $n \geq 2k$, how many such strings are there?

The string must terminate in a 1 or a 0. If it terminates in a 1, then all $k-1$ previous 1's must be immediately followed by 0's (i.e 10 pairs). This leaves $n-1-2(k-1)=n-2k+1$ 0's not immediately following 1's. With $k-1$ 10 pairs and $n-2k+1$ 0's, there are $n-k$ positions to be filled. This can be done in $\binom{n-k}{k-1}$ ways. If the string terminates in a 0, then all k previous 1's must be immediately followed by 0's (i.e 10 pairs). This leaves $n-2k$ 0's not immediately following 1's. With k 10 pairs and $n-2k$ 0's, there are $n-k$ positions to be filled. This can be done in $\binom{n-k}{k}$ ways. The total is then $\binom{n-k}{k-1}+\binom{n-k}{k}$.