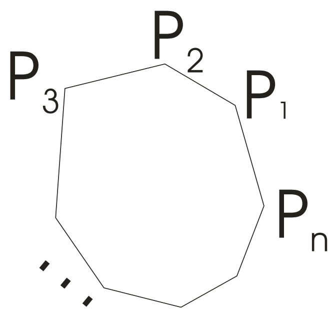
Examination 1 Solutions
CS 336

1. [5] For $n \geq 3$, how many diagonals does a convex polygon with *n* extreme points have? (Consider a convex polygon given by extreme points $< P_1, P_2, \ldots, P_n >$ in counterclockwise order A "diagonal" is a line segment connecting two **non-adjacent** extreme points.)



For each of the *n* extreme points there are *n-3* distinct extreme points that non-adjacent. This would yield $n(n-3)$ endpoints of the diagonals. Since each diagonal has two endpoints, there are $\frac{n(n-3)}{2}$ diagonals of a convex polygon with *n* extreme points.

2. a. [10] Present a combinatorial argument that for all $n \geq 1$:

$$(2n-1)\cdot(2n-3)\cdots 3\cdot 1 = \frac{(2n)!}{n!2^n}$$

Consider the set of all partitions of a set of cardinality $2n$ into *n* pairs. For the left side, begin with any permutation of the $2n$ elements. The first element on the permutation is in some pair and there are $2n-1$ choices for its pair-mate. Removing these two from the permutation, the next element permutation is also in some pair and there are $2n-3$ choices for its pair-mate. The process continues until there are just two elements left in the permutation, and they form the last pair. This yields $(2n-1)\cdot(2n-3)\cdots 3\cdot 1$ different such partitions. Now consider the right hand side. There are $(2n)!$ different permutations of the of the $2n$ elements. Pair the first element with the second, the third with the fourth, etc. This yields a partition into *n* pairs. However, the order among the *n* pairs is irrelevant to the partition and thus for every array of pairs there are 2^n different permutations. Lastly, the order among the pairs, is also irrelevant, so a set of pairs could be arranged in $n!$ different orders. Thus the number of partitions into pairs that ignores order within and among pairs is $\frac{(2n)!}{n!2^n}$ and this must equal $(2n-1)\cdot(2n-3)\cdots 3\cdot 1$.

b. [10] Present a combinatorial argument that for all nonegative integers *k* and *n* satisfying $k \le n-2$

$$\binom{n+2}{k} = \binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2}$$

Let set A have cardinality n and b and c be distinct elements not contained in A. Consider the subsets of $A \cup \{b\} \cup \{c\}$ of cardinality k. For the left hand side, we recognize that $A \cup \{b\} \cup \{c\}$ has cardinality $n+2$, so there are $\binom{n+2}{k}$ such subsets. Alternatively, consider that a subset wither has all k elements coming from A, exactly $k-1$ elements coming from A, or A, exactly $k-2$ elements coming from A. If all k elements come from A, there are $\binom{n}{k}$. If exactly $k-1$ elements come from A, there are $\binom{n}{k-1}$ ways to select those elements and then two choices, b or , to complete the subset. If exactly $k-2$ elements come from A, there are $\binom{n}{k-2}$ ways to select those elements and then both b and must be selected to complete the subset. The total is $\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2}$ and this must equal $\binom{n+2}{k}$.

3. [15] How many partitions are there of a set of 45 elements into a subset of cardinality 3, six subsets of cardinality 4, and three subsets of cardinality 6?

There are $\binom{45}{3}$ ways to select the elements for the subset of cardinality 3.

Removing those leaves 42 elements, and there are $\binom{42}{4\ \ 4\ \ 4\ \ 4\ \ 4\ \ 4}$ ways to select the elements for the six subsets of cardinality 4. However, there is no order amongst these subsets and there are actually $6!$ ways to reorder the subsets, so this has been over counted by a factor of $6!$. Finally, removing those leaves 18 elements, and there are $\binom{18}{6\ \ 6\ \ 6}$ ways to select the elements for the tree subsets of cardinality 6. However, there is no order amongst these subsets and there are $3!$ ways to reorder the subsets, so this has been over counted by a factor of $3!$. The number of such partitions is then $\dfrac{\binom{45}{3}\binom{42}{4\ 4\ 4\ 4\ 4\ 4}\binom{18}{6\ 6\ 6}}{6!3!}$. This can also be written $\dfrac{\binom{45}{3\ 4\ 4\ 4\ 4\ 4\ 4\ 6\ 6\ 6}}{6!3!}$.

4. [15] For $n \geq 1$, what is the value of $\sum_{i_1=0}^{n}(\sum_{i_2=0}^{n-i_1}(\sum_{i_3=0}^{n-i_1-i_2}1))$? Present a combinatorial argument: determine the value of the expression, then defend it by establishing a model and counting it. (Hint: Define $i_4 = n - i_1 - i_2 - i_3$ and then think about putting balls into bins.)

The expression adds one for every ordered triple $< i_1, i_2, i_3 >$ of non-negative integers so that $i_1 + i_2 + i_3 \leq n$. Letting $i_4 = n - i_1 - i_2 - i_3$, this is the same as adding one for every ordered four-tuple $< i_1, i_2, i_3, i_4 >$ of non-negative integers so that $i_1 + i_2 + i_3 + i_4 = n$. That however, is the number of ways of tossing n balls into four bins which is $\binom{n+3}{3}$.

5. [10] Consider strings of four *a*s and four *b*s. Assume all such strings are equally likely. What is the probability that two or more *a*s precede all of the *b*s.

A string has length 8 and is determined by the 4 positions for the *a*s, thus there are $\binom{8}{4}$ equally likely strings. Either exactly two *a*s precede all of the *b*s, exactly three *a*s precede all of the *b*s, or all four *a*s precede all of the *b*s. If exactly two *a*s precede all of the *b*s, the string begins *aab* and there are 5 remaining positions to contain the 2 *a*s. Thus $\binom{5}{2}$ such strings. If exactly three *a*s precede all of the *b*s, the string begins *aaab* and there are 4 remaining positions to contain the last *a*. Thus there are $\binom{4}{1}$ such strings. Lastly there is only one string in which all of the *a*s precede all of the *b*s. In total there are $\binom{5}{2}+\binom{4}{1}+1=15$ strings in which two or more *a*s precede all of the *b*s. The probability of having such a string is $\dfrac{15}{\binom{8}{4}}$

6. [10] Consider rolls of a pair of six-sided dice assuming all possible order pairs of outcomes are equally likely. What is the probability that the sum of values shown on the dice is 8 given that either of the dice shows a 2?

There are $6^2 = 36$ equally likely rolls of the dice. If the first die shows a 2, there are 6 different values for the second die (one of which is 2). If the second die shows a 2, there are 6 different values for the first die (one of which is 2). Thus, 6+6-1=11 different rolls have at least one die with a value of 2. Of these, there are the only way to get a total of 8 is with (2,6) or (6,2). Thus, the probability that the sum of values shown on the dice is 8 given that either of the dice shows a 2 is $\frac{2}{11}$.