Examination 2 Solutions

CS 336

1. [20] Using only Definition 2', prove that the set of finitely long strings using characters from {A, B, C, …, Z, a, b, c, …., z} is infinite**.**

Let S = set of finitely long strings of {A, B, C, …, Z, a, b, c, …., z}. For $s \in S$, define $f : S \to S$ by $f(s) = A \,|\, s$ (i.e. the string consisting of A concatenated with s). For $s, t \in S$, if $s \neq t$, then $f(s) = A \,|\, s \neq A \,|\, t = f(t)$, so f is one-to-one. Let $u = \langle B \rangle$. Since for all $s \in S$ the first element of $f(s)$ is an A, there is no $s \in S$ such that $f(s) = u$. We have then that f maps S into $S \sim \{u\}$, which is a proper subset of S, and by Definition 2', S is infinite.

2. [20] Consider this theorem (that relies upon the Axiom of Choice):

If $f : A \xrightarrow[onto]{} B$, *then there exists a subset* $\hat{A}$ *of* A *such that* $f : \hat{A} \xrightarrow[onto]{1-1} B$.

Use his theorem to prove: If $f : A \xrightarrow[onto]{} B$, and B is infinite then A is infinite.

If $f : A \xrightarrow[onto]{} B$, then there exists a subset $\hat{A}$ of A such that $f : \hat{A} \xrightarrow[onto]{1-1} B$ thus $f^{-1} : B \xrightarrow[onto]{1-1} \hat{A}$. If B is infinite then by Theorem 4 $\hat{A}$ is infinite and by Theorem 3 A is infinite.

3. [20] Is the set of infinitely long strings using characters from {A, B, C, …, Z, a, b, c, …., z} finite, countably infinite, or uncountably infinite**?** Prove your claim.

The set is uncountably infinite. Let $\mathbb{S}$ denote the set of infinitely long strings using characters from {A, B, C, …, Z, a, b, c, …., z} and B denote the set of infinitely long bit strings. Define $f : \mathrm{B} \to \mathbb{S}$ by : $f(s)$ for a bit string s replaces every 0 in the string with the character A and every 1 in the string with the character B. This function is one-to-one since if s and t are bit strings and $s \neq t$, then s and t must differ in some position *k*, (one having a 0 and one having a 1) but then $f(s)$ and $f(t)$ must differ in that same position *k*, (one having an A and one having a B). Thus, $f(s) \neq f(t)$ and f is one-to-one. The set B is uncountably infinite by Theorem and thus, by Theorem , $\mathbb{S}$ is uncountably infinite.

4. [20] Prove that $\sqrt{n^3+1} = o(n^2)$.

Given any $\varepsilon > 0$, let $N = \max\{1, 2/\varepsilon^2\}$. Notice then for $n \geq N$, we have $n \geq 1$ and $n \geq 2/\varepsilon^2$. Thus $\dfrac{2}{n} \leq \varepsilon^2$ and $\dfrac{n^3+1}{n^4} \leq \dfrac{2n^3}{n^4} \leq \varepsilon^2$. So $\dfrac{\sqrt{n^3+1}}{n^2} = \sqrt{\dfrac{n^3+1}{n^4}} \leq \varepsilon$ and $|\sqrt{n^3+1}| = \sqrt{n^3+1} \leq \varepsilon n^2 = \varepsilon\,|n^2|$. Therefore, $\sqrt{n^3+1} = o(n^2)$.

5. **[20]** Employing induction and Theorem 4, prove that for $k \ge 1$, if for $i = 1, 2, \ldots, k$, $f_i = \mathrm{O}(g)$, then $\prod_{i=1}^{k} f_i = \mathrm{O}(g^k)$.

For k=1, we have $\prod_{i=1}^{1} f_i = f_1 = \mathrm{O}(g) = \mathrm{O}(g^1)$ by hypothesis. Now assume $\prod_{i=1}^{k} f_i = \mathrm{O}(g^k)$ and consider $\prod_{i=1}^{k+1} f_i = \prod_{i=1}^{k} f_i \cdot f_{k+1}$. Since $\prod_{i=1}^{k} f_i = \mathrm{O}(g^k)$ and $f_{k+1} = \mathrm{O}(g)$, Theorem 4 guarantees that $\prod_{i=1}^{k+1} f_i = \mathrm{O}(g^k \cdot g) = \mathrm{O}(g^{k+1})$.

6. **[20]** Prove that polynomials are asymptotically dominated by their largest power: That is, for $k \ge 0, \sum_{i=0}^{k} a_i n^i = \mathrm{O}(n^k)$.

By Theorem 6, $0 \le i \le k, n^i = \mathrm{O}(n^k)$. By Theorem 1, $0 \le i \le k, a_i n^i = \mathrm{O}(n^k)$. By Corollary 3.2, $k \ge 0, \sum_{i=0}^{k} a_i n^i = \mathrm{O}(n^k)$.