Examination 2 Solutions

CS 336

1. [20] Using only Definition 2', prove that the set of finitely long bit strings is infinite**.**

Let S = set of finitely long bit strings. For $s \in S$, define $f : S \to S$ by $f(s) = 0 \mid s$ (i.e. the string consisting of 0 concatenated with s). For $s, t \in S$, if $s \neq t$, then $f(s) = 0 \mid s \neq 0 \mid t = f(t)$, so f is one-to-one. Let $u = <1>$. Since for all $s \in S$ the first element of $f(s)$ is an 0, there is no $s \in S$ such that $f(s) = u$. We have then that f maps S into $S \sim \{u\}$, which is a proper subset of S, and by Definition 2', S is infinite.

2. [20] Suppose operating system **S** allows passwords of 6 or more characters from the set $\{A,...,Z,a,...z,0,...,9,\_\}$ and no others. Is the set of legal passwords finite, countably infinite, or uncountably infinite? Prove your claim.

The set is countably infinite. Let P denote the set of finitely long strings using characters from $\{A,...,Z,a,...z,0,...,9,\_\}$ and for $n \geq 6$, P_n be the set of strings using characters from $\{A,...,Z,a,...z,0,...,9,\_\}$ of length n. Each set P_n is finite (in fact, having cardinality 63^n) and $\mathrm{P} = \bigcup_{n=6}^{\infty} \mathrm{P}_n = \bigcup_{n \in \mathbb{N}} \mathrm{P}_{n+6}$ .Since P is the countably infinite union of finite sets, by Theorem 9, it is countable. Finally consider the mapping $f : \mathbb{N} \to \mathrm{P}$ defined by $f(n) = < A...A >$ having length $n+6$ for $n \in \mathbb{N}$. This function is clearly one-to-one since if $n \neq m$, $f(n) \neq f(m)$ since they have different lengths. By Theorem 4, P is infinite, hence countably infinite.

3. [20] Is the set of circles in the plane finite, countably infinite, or uncountably infinite**?** Prove your claim.

The set is uncountably infinite. Let $\mathbb{C}$ denote the set of circles in the plane and consider $f : [0,1] \to \mathbb{C}$ defined by $f(x)$ is the circle or radius 1 with center $(x,0)$. This function is one-to-one since if x and y are elements of $[0,1]$ and $x < y$, then the circle $f(x)$ contains the point $(x-1,0)$ but the circle $f(y)$ does not contain that point since the distance from $(x-1,0)$ to $(y,0)$ is $y-x+1$ which is greater than Thus, f is one-to-one and since the interval $[0,1]$ is uncountably infinite, by Theorem 11, the set $\mathbb{C}$ is uncountably infinite.

4. [20] Using no other asymptotic dominance theory than definitions, prove that $6n^{7/8} + 5n^{3/2} = O(n^2)$.

Let $M = 11$ and $N = 1$. For $n \geq N$, we have $n^{7/8} \leq n^{3/2} \leq n^2$, so $|6n^{7/8} + 5n^{3/2}| = 6n^{7/8} + 5n^{3/2} \leq 6n^2 + 5n^2 = 11n^2 = M|n^2|$. Therefore, $6n^{7/8} + 5n^{3/2} = O(n^2)$.

5. [20] Employing induction prove that for $k \geq 1$, if for $i = 1, 2, \ldots, k$, $f_i = O(f_{i+1})$, then $f_1 = O(f_{k+1})$.

For k=1, we have $f_1 = O(f_2)$ thus $f_1 = O(f_2) = O(f_{k+1})$. Let us assume the result is true for some $k \geq 1$, and attempt to prove that if for $i = 1, 2, ..., k+1$, $f_i = O(f_{i+1})$, then $f_1 = O(f_{k+2})$. By the inductive hypothesis we have $f_1 = O(f_{k+1})$ and we also know $f_{k+1} = O(f_{k+2})$. By definition, there exist $M, \bar{M}, N,$ and $\bar{N}$, so that for $n \geq N, |f_1(n)| \leq M|f_{k+1}(n)|$ and .for $n \geq \bar{N}, |f_{k+1}(n)| \leq \bar{M}|f_{k+2}(n)|$. Thus for $n \geq \bar{\bar{N}} = \max\{N, \bar{N}\}, |f_1(n)| \leq M|f_{k+1}(n)| \leq M\bar{M}|f_{k+2}(n)|$, so $f_1 = O(f_{k+2})$.

6. [20] Prove that $2^n = o(n!)$. (Hint: $\prod_{i=1}^{n} \frac{2}{i} = \prod_{i=1}^{3} \frac{2}{i} \cdot \prod_{i=4}^{n} \frac{2}{i} = \frac{4}{3} \prod_{i=4}^{n} \frac{2}{i}$ and $\frac{2}{i} \leq \frac{1}{2}$ for $i \geq 4$.)

Given any $\varepsilon > 0$, let $N = \max\{1, \log_2 \frac{32}{3\varepsilon}\}$. Notice then for $n \geq N$, we have

$2^n \geq \frac{32}{3\varepsilon} = \frac{4}{3}\frac{8}{\varepsilon}$, so

$$\varepsilon \geq \frac{4}{3} 8 \left(\frac{1}{2}\right)^n = \frac{4}{3}\left(\frac{1}{2}\right)^{n-3} = \frac{4}{3}\prod_{i=4}^{n}\frac{1}{2} \geq \frac{4}{3}\prod_{i=4}^{n}\frac{2}{i} = \prod_{i=1}^{n}\frac{2}{i} = \frac{\prod_{i=1}^{n} 2}{\prod_{i=1}^{n} i} = \frac{2^n}{n!}.$$

and $|2^n| = 2^n \leq \varepsilon n! = \varepsilon|n!|$. Therefore, $2^n = o(n!)$.